De Moivre's Theorem

A Literature and Curriculum Project on
Roots, Powers, and Complex Numbers

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<u>Abstract</u>

Complex numbers and their basic operations are important components of the college-level algebra curriculum. Common learning objectives of college algebra are the computation of roots and powers of complex numbers, and the finding of solutions to equations that have complex roots. A portion of this instruction includes the conversion of complex numbers to their polar forms and the use of the work of the French mathematician, Abraham De Moivre, which is De Moivre's Theorem.

The intent of this research project is to explore De Moivre's Theorem, the complex numbers, and the mathematical concepts and practices that lead to the derivation of the theorem. The research portion of this document will a include a proof of De Moivre's Theorem,

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

where $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, the application of this theorem, nth roots, and roots of unity, as well as related topics such as Euler's Formula:

$$e^{ix} = \cos x + i\sin x,$$

and Euler's Identity $e^{i\pi} + 1 = 0$.

This research will provide a greater understanding of the deeper mathematical concepts necessary to effectively teach the subject matter. In addition it will provide the opportunity to explore lessons and activities that will facilitate students developing a greater appreciation for the significance and power of the complex number system.

Table of Contents

Part One: The History and Mathematics of De Moivre's Formula

Part Two: Teaching Complex Numbers and De Moivre's Formula

Part One:

The History and Mathematics of De Moivre's Formula

Chapter 1 – Some History

Section 1.1 – History of the Complex Numbers

The set of complex or imaginary numbers that we work with today have the fingerprints of many mathematical giants.

In 1545 Gerolamo Cardano, an Italian mathematician, published his work *Ars Magnus* containing a formula for solving the general cubic equation

$$x^3 + ax^2 + bx + c = 0$$

While deriving the formula, Cardano came across the solution with the square root of a negative number. Cardano did not publish this *casus irreducibilis,* considering it useless.

Rafael Bombelli introduced a label for such numbers in his set of books *l'Algebra* published in 1572 and 1579. While Cardano chose not to publish his work with complex numbers, Bombelli found the *casus irreducibilis* had validity and introduced a notation, calling it a *"piu di meno,"* for $+\sqrt{-1}$ and "*meno di meno"* for $-\sqrt{-1}$. He devised a table to explain his notation.

piu di meno via piu di meno fa meno

piu di meno via meno di meno fa piu

Meno di meno via piu di meno fa piu

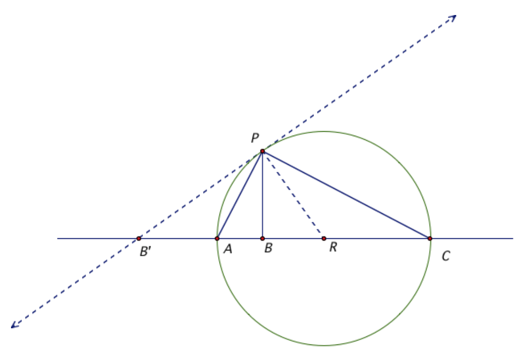
meno di men via meno di meno fa meno (Bashmakovia & Smirnova, 2000).

Which means:

$$
\begin{aligned}
\sqrt{-1} \times \sqrt{-1} &= -1 \\
\sqrt{-1} \times \left(-\sqrt{-1}\right) &= +1 \\
\left(-\sqrt{-1}\right) \times \sqrt{-1} &= +1 \\
\left(-\sqrt{-1}\right) \times \left(-\sqrt{-1}\right) &= -1.
\end{aligned}
$$

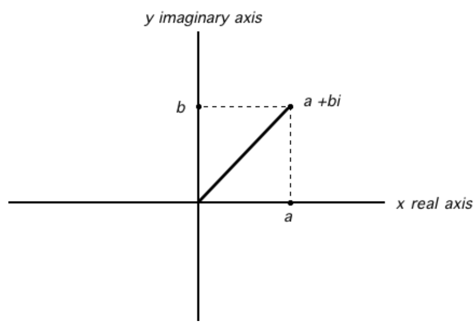
Abraham De Moivre (1667 – 1754) further extended the study of such numbers when he published *Miscellanea Analytica* in 1730, utilizing trigonometry to represent powers of complex numbers. His work is the subject of the mathematical portion of this paper, and his life is described in more detail in the next section.

John Wallis contributed to the visualization of complex numbers in a treatise titled *Algebra*. Employing a single axis with positive values to the right and negative values to the left, Wallis constructed a circle with one end of the diameter $\overline{AC}$ at the origin and the other to the right as a positive value. By then constructing similar right triangles within and about the circle and tangent to the circle, he reasoned the geometric mean would hold true regardless of positive or negative values assigned to vertices.



That is $\dfrac{AB}{x} = \dfrac{x}{PB}$ for side lengths AB and PB or $x = \sqrt{AB \cdot PB}$ regardless of the position relative to the axis. While Wallis' theories furthered the geometric image of complex numbers, they were awkward and inconclusive (Nahin, 1998).

The Norwegian surveyor Caspar Wessel presented his visual interpretation of complex numbers to the Royal Danish Academy of Science in 1797. Wessel described a complex number $a + bi$, as point (a, b) on a plane consisting of a real axis and an imaginary axis (Nahin, 1998).



Sources attribute other brilliant men working in the field of mathematics, during the same time period, with utilizing similar representations of complex numbers. Carl Fredrick Gauss (1777 – 1855) relied upon a positional description of $\sqrt{-1}$, much like Wessel's. According to Paul J. Nahin, "Gauss had been in possession of these concepts in 1796 and had used them to reproduce without Gauss' knowledge, Wessel's results" (1998, p. 82). In 1799, as part of his dissertation, Gauss relied on this knowledge to prove that any polynomial with real coefficients could be written as the product of linear or quadratic factors. Any such polynomial would then have a solution contained in the set of complex numbers (Mazur, 207). This we now know as the Fundamental Theorem of Algebra. Gauss is also attributed with the introduction of the term *complex number*.

Leonhard Euler (1707 – 1783), a Swiss mathematician, refined the geometric definition of complex numbers. He described the solutions of the equation $x^n - 1 = 0$ as vertices of a regular polygon in the plane. Euler also introduced the notation $\sqrt{-1} = i$. He defined the complex exponential, and published his proof of the identity $e^{ix} = \cos x + i \sin x$, in 1748 (Nahin, 1998).

Isaac Newton once said, “If I have seen further it is by standing on ye shoulders of Giants” (Livio 101). While Newton’s word where likely a slight to his contemporary Robert Hooke, it is certain that our understanding of complex numbers, or any other mathematical concept, are an evolution of the contributions of many.

Section 1.2 – History of Abraham De Moivre

Abraham De Moivre was born in Champagne France on May 26, 1667. He became interested in mathematics at an early age and pursued mathematics intentionally in school and on his own. It was De Moivre’s unfortunate luck to be born into a protestant family at a time when the ruling monarchy was restricting religious freedom. He left France when he was 18, to live in London where his luck would most decidedly change. He was eventually thrown into the company of many brilliant mathematicians. Throughout his time in London he supported himself as a tutor.

Shortly after his arrival in London, De Moivre obtained a copy of Isaac Newton’s book *Principia.* He studied Newton’s work intently even tearing out, and carrying pages of the book so that he could study the work during spare moments. Tradition has it that De Moivre eventually became such an expert on Newton’s work that Newton himself would refer questions regarding *Principia* to de Moivre, saying, “he knows more about it than I do” (Nahin 1998).

De Moivre met Edmond Halley in 1692. Halley took a paper written by De Moivre to the Royal Society. Through this introduction De Moivre became part of the exclusive society where men like Newton, Halley, Wallis and Cotes exchanged and clashed over ideas that were to become the many of the founding precepts of mathematical theory today. He was elected to the Royal Society in 1697. He was appointed to a commission in 1712 that would settle the battle between Newton and Leibniz over the right to claim himself as the inventor of calculus. He of course ruled in favor of Newton (Maor 1998).

De Moivre made many contributions to the field of mathematics, mainly in the areas of theory of probability and algebra/trigonometry. In 1718 he published *The Doctrine of Chances: or, a Method of Calculating the Probability of Events in Play.* In 1725 De Moivre published a work *A Treatise of Annuities upon Lives*, an examination of mortality statistics. De Moivre published a formula in 1733 that approximated n factorial, $n! \approx cn^{n+1/2}e^{-n}$, where c is some constant. Unfortunately De Moivre was not able to determine the value of c. Today the formula is known as Stirling's formula, since James Stirling of Scotland determined the value to be $n! \approx \sqrt{2\pi n}\, n^n e^{-n}$ (Maor 1998).

De Moivre's third publication in 1730 is the work that motivates this research, *Miscellanea Analytica.* Here De Moivre tackles the important dilemma of the time, the factorization of the polynomial $x^{2n} + px^n + 1$ into quadratics. De Moivre was continuing the efforts of Roger Cotes. Essential to his work was a trigonometric representation of powers of complex numbers, we know today as De Moivre's Theorem. That is $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$. It is interesting to note that it was Euler and not De Moivre that wrote this result explicitly (Nahin 1998).

Despite De Moivre's mathematical contributions, he continued to support himself by tutoring. He was never able to attain an appointment to a chair at a university. In his latter years he began to sleep more and more. It is reported that he predicted the day of his own death. After observing his sleep time increased each day by an additional 15 minutes he calculated the arithmetic progression until he would sleep forever. His calculations were correct. He died November 27, 1754 (Maor, 1998).

Chapter 2 – Some Mathematics

Section 2.1 – Complex Numbers—Rectangular Form

The standard form of a complex number is a + b*i* where *a* is the real part of the number and *b* is the imaginary part, and of course we define $i = \sqrt{-1}$. Also we assume $i^2 = -1$ since $\left(\sqrt{-1}\right)^2 = -1$.

The set of complex numbers contains the set of all real numbers, that is when b = 0.

We apply the same properties to complex numbers as we do to real numbers. To be considered equal, two complex numbers must be equal in both their real and their imaginary components. That is to say, the numbers $a + bi = c + di$ are equal to one another if and only if $a = c$ and $b = d$.

Complex numbers have the same additive identity as the real number system, namely zero. The additive inverse of the complex number a + b*i* is $-(a+bi) = -a-bi$ thus $(a+bi)+(-a-bi) = 0+0i = 0$.

When we add or subtract complex numbers, we add or subtract the real parts and the imaginary parts separately. Given the complex numbers a + b*i* and c + d*i*, we add or subtract as follows:

$$(a+bi)+(c+di) = (a+c)+(b+d)i$$

$$(a+bi)-(c+di) = (a-c)+(b-d)i.$$

For example: $(2+2i)+(4+5i) = (6+7i)$

$$(2+2i)-(4+5i) = (-2-3i)$$

It is important to note that the sum or difference of two complex numbers can become a real number with no imaginary part,

$$(1+6i)-(5+6i)=(-4+0i)=-4\,.$$

Because the complex numbers contain the set of real numbers, however, this fact does not contradict the fact that the complex numbers are closed under both addition and multiplication. Many other properties of real numbers apply to complex numbers:

- The Associative Property of Addition

$$[(a+bi)+(c+di)]+(e+fi)=(a+bi)+[(c+di)+(e+fi)]$$

- The Commutative Property of Addition

$$(a+bi)+(c+di)=(c+di)+(a+bi)$$

- The Distributive Property of Multiplication over Addition

$$k(a+bi)=(a+bi)+(a+bi)+(a+bi)+\ \ldots\ +(a+bi)=(ka+kbi) \quad \text{for } k\in\mathbf{R}.$$

These same properties hold for multiplication of complex numbers. Here we must rely on the defined value $i^2=-1$. Then for the complex numbers a + bi and c + di,

$$\begin{aligned}(a+bi)(c+di)&=a(c+di)+bi(c+di)\\&=ac+(ad)i+(bc)i+(bd)i^2\\&=ac+(ad)i+(bc)i+(bd)(-1)\\&=ac-bd+(ad)i+(bc)i\\&=(ac-bd)+(ad+bc)i\end{aligned}$$

As the above computation illustrates, we can also use the polynomial multiplication process commonly called FOIL to multiply complex numbers. In order to find the quotient of complex numbers we rely on the complex conjugate. The complex conjugate of $(a+bi)$ is the complex number $(a-bi)$, where the imaginary parts differ only by a sign and the product of the two is a real number.

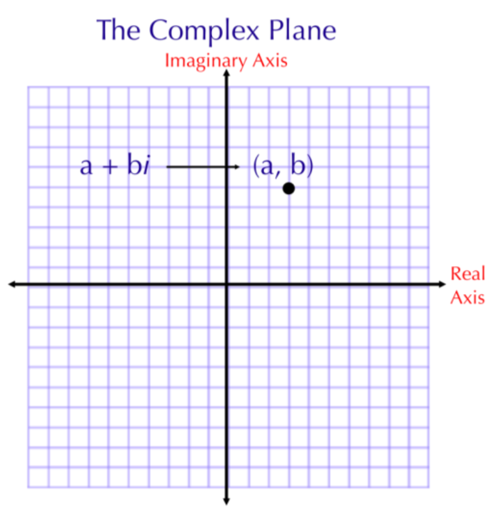
$$\begin{aligned}(a+bi)(a-bi) &= a(a-bi)+bi(a-bi)\\ &= aa-(ab)i+(ab)i-(bb)i^2\\ &= a^2-b^2(-1)\\ &= a^2+b^2\end{aligned}$$

Then for the quotient of complex numbers we have

$$\begin{aligned}\frac{a+bi}{c+di} &= \left(\frac{a+bi}{c+di}\right)\left(\frac{c-di}{c-di}\right)\\ &= \frac{ac-adi+bci-bdi^2}{c^2+d^2}\\ &= \frac{ac-adi+bci-bd(-1)}{c^2+d^2}\\ &= \frac{ac-adi+bci+bd}{c^2+d^2}\\ &= \frac{(ac+bd)+(bc-ad)i}{c^2+d^2}\end{aligned}$$

where the simplified value has no imaginary part in the denominator. Notice that, as a result, the complex numbers are closed under division, as long as the divisor is nonzero.

We graph complex numbers on the coordinate system called the 'complex plane', where the horizontal axis is the real axis and the vertical axis is the imaginary axis. On the complex plane, every ordered pair or point (a, b) corresponds to a unique complex number a +b*i*.



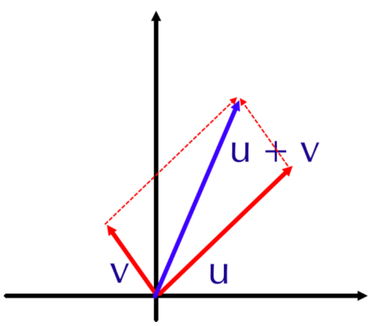
The absolute value of a complex number is defined as the distance from the origin to the ordered pair or point (a, b),

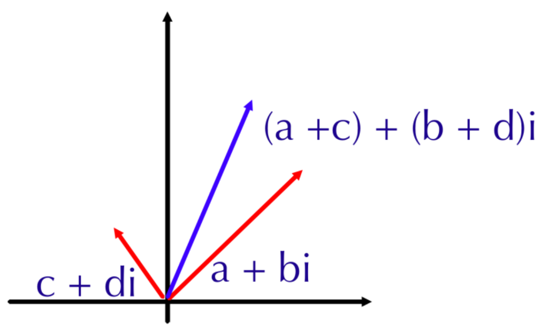
$$|a+bi| = \sqrt{a^2+b^2}\,.$$

If we construct a segment connecting any complex number on the plane with the origin, then we will quickly observe that complex number operations share many similarities to operations of vectors. The absolute value of a complex number is the same calculation as the magnitude of a vector.

$$|a+bi| = \sqrt{a^2+b^2} \qquad\qquad \|\mathbf{v}\| = \sqrt{{v_1}^2+{v_2}^2}$$

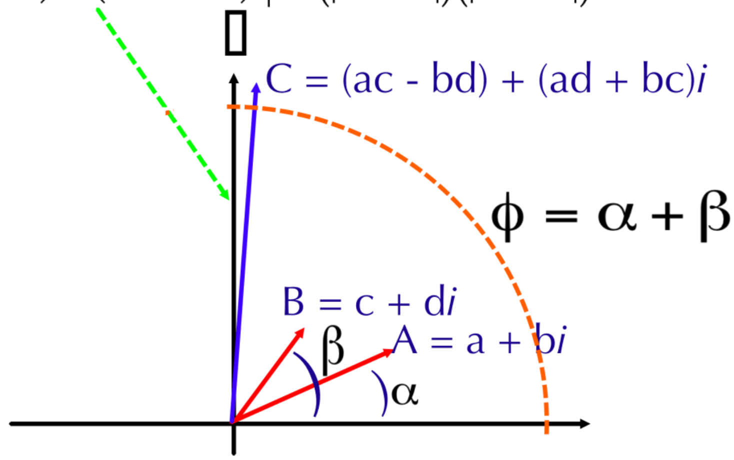
When we add and subtract complex numbers graphically, it appears very much like vector addition and subtraction.





While the multiplication of complex numbers does not match the process for any vector operation, if we visualize the complex numbers as vectors, it is much easier to understand what is occurring geometrically. On the graph below we see that the multiplication of complex numbers $A = a+bi$ and $B = c+di$ produces a new complex number $C = (ac-bd)+(ad+bc)i$, with magnitude equal to the product of the magnitudes of A and B. In addition, the angle formed by the positive x-axis and $C = (ac-bd)+(ad+bc)i$ is equal to the sum of the angles formed by each of the complex numbers $A = a+bi$ and $B = c+di$, and the positive x-axis.

$$\left|(ac-bd)+(ad+bc)i\right| = \left(|a+bi|\right)\left(|c+di|\right)$$



Operations with complex numbers can often become tedious and lengthy. Further discussion of the multiplication and division of complex numbers necessitates the consideration of another form of complex numbers.

Section 2.2 – Complex Numbers—Polar Form

We can represent a complex number using trigonometry much like we represent vectors in trigonometric form. We also call this representation the 'polar form' of complex numbers. Rather than using a coordinate for the real part and the imaginary part, we use the absolute value of the complex number and the directed angle from the positive x-axis or polar axis to the line segment connecting the complex point to the pole, measured in a counter-clockwise direction.

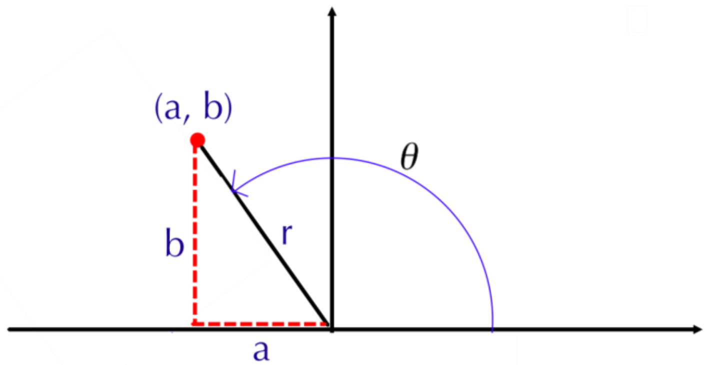
Then the parameters of the rectangular and polar form are related as follows:

$$a = r\cos\theta \qquad \text{and} \qquad b = r\sin\theta$$

with

$$r = \sqrt{a^2+b^2}, \quad \text{and} \quad \tan\theta = \left(\frac{b}{a}\right)$$

so that $$z = a + bi = (r\cos\theta) + (r\sin\theta)i..$$



Here r is called the modulus of the z and θ is called the argument. Unlike the rectangular coordinates (a, b) for a complex number, the ordered pair (r, θ) is not unique since for any angle

$$\cos\theta = \cos(\theta + 2\pi k) \quad \text{and} \quad \sin\theta = \sin(\theta + 2\pi k).$$

Generally, to address this issue, we restrict such representations of complex numbers to an interval such as $0 \le \theta \le 2\pi$, although a negative angle value may be used. For example we can find the trigonometric representation of the number $z = -2 - 2i$ in the following manner:

$$r = \sqrt{(-2)^2 + (-2)^2} = \sqrt{8} = 2\sqrt{2}$$

$$\tan\theta = \frac{-2}{-2} = 1 \text{ and}$$

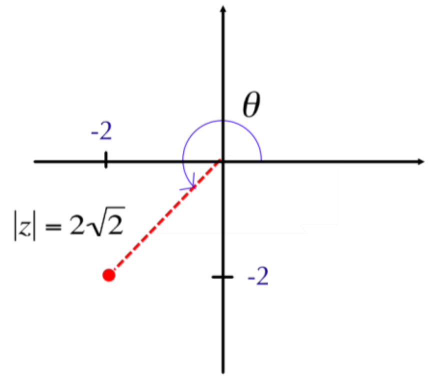
$$\theta = \tan^{-1}(1) = 45°$$

but we know that z lies in the 3rd quadrant and arctangent has a range of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ so

$$\theta = 45° + 180° = 225°, \quad \text{or} \quad 225° = \frac{5\pi}{4}$$

Putting it all together, we have

$$z = 2\sqrt{2}(\cos 225° + i\sin 225°) \qquad \text{or} \qquad z = 2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right).$$



We use trigonometric or polar form of imaginary numbers for several calculations including multiplication and division of complex numbers, and for finding powers of complex numbers.

Given the complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$ we find the product

$$z_1 z_2 = r_1 r_2(\cos\theta_1 = i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= r_1 r_2(\cos\theta_1\cos\theta_2 + \cos\theta_1 i\sin\theta_2 + i\sin\theta_1\cos\theta_2 + i^2\sin\theta_1\sin\theta_2)$$

$$= r_1 r_2[\cos\theta_1\cos\theta_2 + (-1)\sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1 r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

Here we see the formula for the cosine of the sum of two angles and the sine of the sum of two angles thus

$$z_1 z_2 = r_1 r_2[\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)].$$

We have a very easy calculation to find the product of two complex numbers. We simply multiply the modulii and add the arguments. This method is a much more efficient model of the previously mentioned graphical representation of complex number multiplication.

Given the same two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = (r_2\cos\theta_2 + i\sin\theta_2)$ we find the quotient as follows.

$$
\begin{aligned}
\frac{z_1}{z_2} &= \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)} \\
&= \frac{r_1(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 - i\sin\theta_2)}{r_2(\cos\theta_2 + i\sin\theta_2)(\cos\theta_2 - \cos\theta_2)} \\
&= \frac{r_1\left[\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2\right]}{r_2\left[(\cos\theta_2)^2 - i\cos\theta_2\sin\theta_2 + i\cos\theta_2\sin\theta_2 - i^2(\sin\theta_2)^2\right]} \\
&= \frac{r_1\left[\cos\theta_1\cos\theta_2 - (-1)\sin\theta_1\sin\theta_2 + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right]}{r_2\left[\cos^2\theta_2 - (-1)\sin^2\theta_2\right]} \\
&= \frac{r_1\left[(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)\right]}{r_2\left[\cos^2\theta_2 + \sin^2\theta_2\right]}
\end{aligned}
$$

Here we see the formula for the cosine of the difference of two angles and the sine of the difference of two angles, and a Pythagorean identity thus

$$\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right].$$

Then for the quotient of two complex numbers we find the quotient of the modulii and the difference of the arguments.

Section 2.3 – Powers of Complex Numbers—De Moivre’s Theorem

In order to compute powers of complex numbers we must consider the process of repeated multiplication. Given $z = r(\cos\theta + i\sin\theta)$, then

$$z^2 = [r(\cos\theta + i\sin\theta)][r(\cos\theta + i\sin\theta)] = r^2(\cos 2\theta + i\sin 2\theta)$$
$$z^3 = [r^2(\cos 2\theta + i\sin 2\theta)][r(\cos\theta + i\sin\theta)] = r^3(\cos 3\theta + i\sin 3\theta)$$
$$z^4 = [r^3(\cos 3\theta + i\sin 3\theta)][r(\cos\theta + i\sin\theta)] = r^4(\cos 4\theta + i\sin 4\theta)$$
$$z^5 = [r^4(\cos 4\theta + i\sin 4\theta)][r(\cos\theta + i\sin\theta)] = r^5(\cos 5\theta + i\sin 5\theta)$$

As we continue to increase the power of z, we can see a pattern developing. This pattern is the core of the theorem named after the French mathematician Abraham De Moivre.

De Moivre's Theorem: If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, then,

$$z^n = [r(\cos\theta + i\sin\theta)]^n = r^n(\cos n\theta + i\sin n\theta).$$

Using this theorem we can easily compute the power of a complex number such as $z = (2+2i)$. First we must convert the complex number to its polar form:

$$z = (2+2i) = 2\sqrt{2}(\cos 45° + i\sin 45°),$$

with

$$r = \sqrt{2^2 + 2^2} = \sqrt{8} = 2\sqrt{2}\text{, and } \tan^{-1}\left(\frac{2}{2}\right) = 45°,$$

where z lies in the1st quadrant.

Then

$$z^6 = (2+2i)^6 = [2\sqrt{2}(\cos 45° + i\sin 45°)]^6 = (2\sqrt{2})^6(\cos 270° + i\sin 270°) = -512i.$$

Section 2.4 – The Proof of De Moivre's Theorem

To prove De Moivre's Theorem, we use a simple proof by induction. Given a complex number,

$$z = (\cos\theta + i\sin\theta)$$

we can easily show using repeated multiplication that for n = 0, 1, 2, 3, and 4,

$$z^0 = \left[r^0(\cos 0\theta + i\sin 0\theta)\right] = 1(\cos 0 + i\sin 0) = 1 + i0 = 1$$

$$z^1 = \left[r^1(\cos\theta + i\sin\theta)\right]^1 = r(\cos\theta + i\sin\theta)$$

$$z^2 = \left[r^2(\cos\theta + i\sin\theta)\right]^2 = \left[r(\cos\theta + i\sin\theta)\right]\left[r(\cos\theta + i\sin\theta)\right] = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^3 = \left[r^3(\cos\theta + i\sin\theta)\right]^3 = \left[r^2(\cos 2\theta + i\sin 2\theta)\right]\left[r(\cos\theta + i\sin\theta)\right] = r^3(\cos 3\theta + i\sin 3\theta)$$

$$z^4 = \left[r^4(\cos\theta + i\sin\theta)\right]^4 = \left[r^3(\cos 3\theta + i\sin 3\theta)\right]\left[r(\cos\theta + i\sin\theta)\right] = r^4(\cos 4\theta + i\sin 4\theta)$$

Now let us assume that $z^n = \left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$

is true for some $n \in \mathbf{Z}^+$.

Then we must show that this implies it is true for all *n + 1,* that is,

$$\left[r(\cos\theta + i\sin\theta)\right]^{n+1} = r^{n+1}\left(\cos(n+1)\theta + i\sin(n+1)\theta\right).$$

Then given

$$\left[r(\cos\theta + i\sin\theta)\right]^n = r^n(\cos n\theta + i\sin n\theta)$$

we multiply both sides of the equation by $\left[r(\cos\theta + i\sin\theta)\right]$.

Then

$$\left[r(\cos\theta + i\sin\theta)\right]\left[r(\cos\theta + i\sin\theta)\right]^n = \left[r^n \cos n\theta + i\sin n\theta\right]\left[r(\cos\theta + i\sin\theta)\right]$$

Therefore

$$\left[r(\cos\theta + i\sin\theta)\right]^{n+1} = r^n r\left[\cos n\theta\cos\theta + \cos n\theta i\sin\theta + i\sin n\theta\cos\theta - \sin n\theta\sin\theta\right].$$

We then employ the use of the common trigonometric formulas for the sum of an angle for sine and cosine,

$$\sin(x+y) = \sin x \cos y + \cos x \sin y \quad \text{and} \quad \cos(x+y) = \cos x \cos y - \sin x \sin y.$$

We let $x = n\theta$, and $y = \theta$ and we have

$$r^{n+1}\left[\cos(n\theta+\theta)+i\sin(n\theta+\theta)\right] = r^{n+1}\left[\cos(n+1)\theta + i\sin(n+1)\theta\right],$$

as desired for all positive integers.

We must also consider $n \in \mathbf{Z}^-$ for

$$z^{-n} = \left[r(\cos\theta + i\sin\theta)\right]^{-n} = r^{-n}\left[\cos(-n\theta)+i\sin(-n\theta)\right].$$

Since cosine and sine are even and odd functions respectively, we have

$$\begin{aligned}\cos(-n\theta)+i\sin(-n\theta) &= \cos(n\theta)-i\sin(n\theta)\\ &= \frac{\cos(n\theta)-i\sin(n\theta)}{\cos^2(n\theta)+\sin^2(n\theta)}\\ &= \frac{1}{\cos(n\theta)+i\sin(n\theta)}\cdot\frac{1}{\cos(n\theta)-i\sin(n\theta)}\cdot\cos(n\theta)-i\sin(n\theta)\\ &= \frac{1}{\cos(n\theta)+i\sin(n\theta)}\end{aligned}$$

Therefore

$$r^{-n}\left[\cos(-n\theta)+i\sin(n\theta)\right] = \frac{1}{r^n}\left\lfloor\frac{1}{\cos n\theta + i\sin n\theta}\right\rfloor.$$

Chapter 3 – Some Uses and Related Content

Section 3.1 – Extracting Roots

Potentially the greatest value of De Moivre's work lies in the ability to find the n distinct roots of a complex number. If we let $z = p(\cos\theta + i\sin\theta)andz^n = w$,

then for $w = r(\cos\varphi + i\sin\varphi)$ where $z^n = [p(\cos\theta + i\sin\theta)]^n$

we have $p^n(\cos n\theta + i\sin n\theta) = r(\cos\varphi + i\sin\varphi)$.

So that implies that $p^n = r$ and $n\theta = \varphi$, or $p = \sqrt[n]{r}$ and $\theta = \frac{\varphi}{n}$.

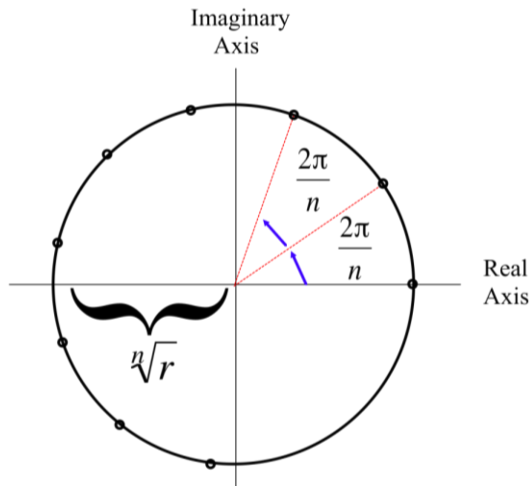
Since both cosine and sine have a period of 2π, we have solutions to both sides of the equation

$n\theta = \varphi$, that is $n\theta = \varphi + 2\pi k$ or $\theta = \frac{\varphi + 2\pi k}{n}$ with $k = 0, 1, 2, \ldots, n-1$.

If we let $k = n$ then we repeat the solutions since $\theta = \frac{\varphi}{n}$ and $\frac{\varphi + 2\pi k}{n} = \frac{\varphi}{n} + 2\pi$ are co-terminal angles. Therefore for the positive integer *n*, we find *n* distinct nth roots of the complex number

$w = r(\cos\varphi + i\sin\varphi)$ by $z = \sqrt[n]{r}\left(\cos\frac{\varphi + 2\pi k}{n} + i\sin\frac{\varphi + 2\pi k}{n}\right)$.

Visually we see, on the complex plane, these solutions or nth roots lie on a circle of radius $n\sqrt{r}$, with n solutions evenly spaced at $\frac{2\pi}{n}$ intervals.

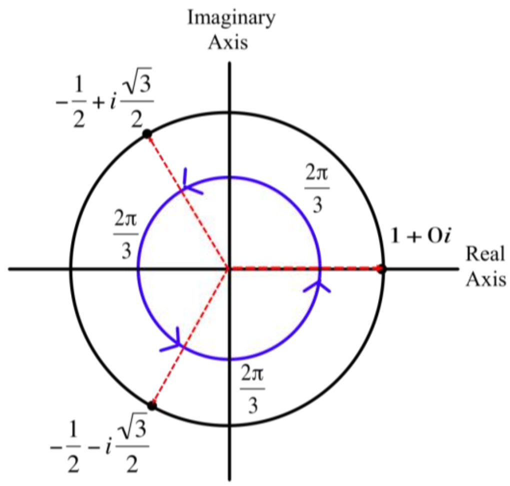


Using this formula we can easily compute the third roots of 1. First we represent 1 as a complex number, that is 1 = 1+0i. Then the modulus $r=\sqrt{1^2+0^2}=1$, and the argument $\theta=\tan^{-1}\left(\frac{0}{1}\right)=0$, so we have $1=\cos 0+i\sin 0$.

Then for the third roots of 1 we have $\cos\frac{0+2\pi k}{3}+i\sin\frac{0+2\pi k}{3}$ with k = 0, 1, and 2.

Then the roots are as follows,

$$\cos 0+i\sin 0=1$$
$$\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$$
$$\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}=-\frac{1}{2}-i\frac{\sqrt{3}}{2}$$



For any set of nth roots of 1, the n distinct roots are called the nth roots of unity. These roots will lie on the unit circle, as seen above, and complex solutions will occur as conjugate pairs.

Section 3.2 – Power Series and Euler's Formula

Most any pre-calculus text will contain a chapter of sequences and series. As part of the unit of sequences and series, students become familiar with arithmetic, geometric and power series. Two familiar power series are those used to represent sine and cosine and the number *e*.

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ \ldots$$

$$\cos x = 1 - \frac{x^2}{21} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ \ldots$$

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ \ldots$$

Leonard Euler (1707 – 1783), a Swiss mathematician, derived a formula relating the three series. Euler's formula is $e^{ix} = \cos x + i \sin x$,

where $e^{ix} = 1 + \frac{ix}{1!} + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ \ldots \ + \frac{(ix)^{n-1}}{(n-1)!} + \ \ldots$

$$= 1 + ix + \frac{i^2x^2}{2!} + \frac{i^3x^3}{3!} + \frac{i^4x^4}{4!} + \ \ldots \ + \frac{i^{(n-1)}x^{(n-1)}}{(n-1)!} + \ \ldots$$

$$= 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \frac{ix^5}{5!} - \frac{x^6}{6!} - \frac{ix^7}{7!} + \ \ldots$$

Now if we group real terms and imaginary terms we have

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ \ldots\right) + \left(ix - \frac{ix^3}{3!} + \frac{ix^5}{5!} - \frac{ix^7}{7!} + \ \ldots\right)$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ \ldots\right)$$

$$= \cos x + i \sin x$$

If we let $x = \pi$, then

$$e^{i\pi} = \cos \pi + i \sin \pi = -1 + i0$$

or

$$e^{i\pi} + 1 = 0,$$

an equation relating the five most important numerical constants in mathematics. This relationship encourages one to explore the question, “How many mathematicians does it take to screw in a light bulb?” The answer: $-e^{\pi i}$ which of course equals 1 (Weisstein).

Part Two:

Teaching Complex Numbers and De Moivre's Formula

Overview of the Curriculum

The following lessons were designed for Pre-calculus classes taught at Oregon City High School (OCHS) in Oregon City, Oregon. This class is a two-trimester class, Pre-Calculus A and Pre-Calculus B, with a maximum of 35 students enrolled in each class. The student population consisted of primarily juniors and seniors with just a few sophomores. At OCHS students are required to earn a C or higher in Algebra 2 before they can take Pre-Calculus. Students are also strongly encouraged to take a one-trimester Trigonometry class before taking Pre-Calculus B but it is not required. As one of the two instructors teaching Pre-Calculus at OCHS, I taught 3 sections of Pre-Calculus A during fall trimester, and 3 sections of Pre-Calculus B during winter trimester. Based on various influences—scheduling, failures, student preference—the student roster changed from fall to winter with about 60% of the students I taught in the fall returning for Pre-Calculus B in the winter.

The student population of Pre-Calculus A and B comes with a wide variety of skill levels. During the first few weeks of Pre-Calculus A students spend time reviewing topics from Algebra 2 including graphing equations by hand and with the calculator, solving equations graphically and algebraically, and solving inequalities graphically and algebraically. It is not unusual to find students who are unfamiliar or have great difficulty with point-slope form of a line, factoring to solve, completing the square to solve, and the domain and range of functions. In Pre-Calculus B we have students who have taken Trigonometry and can solve right triangles with confidence while other students struggle to make sense of SOH CAH TOA. It can be challenging to find meaningful lessons and activities that meet the needs of this diverse group of learners.

The following lessons are taken from the textbook PRECALCULUS WITH LIMITS A GRAPHING APPROACH, Third Edition. Ron Larson, Robert P. Hostetler, Bruce H. Edwards. The lessons were taught over the two-trimester time period but not as one continuous unit. There is a curriculum map already in place at OCHS and within the math department there are strong feelings towards maintaining the pacing calendar and section order developed by previous instructors.

For this research project Lesson 1, Complex Numbers, and Lesson 2, The Fundamental Theorem of Algebra, were taught during Pre-Calculus A, during the sixth or seventh week of the trimester. They were lessons from the second chapter of the textbook titled *Polynomials and Rational Functions*. Lesson 3, The Complex Plane, was taught in Pre-Calculus B during the seventh or eighth week of the winter trimester, and was an extension of the Complex Numbers section from the second chapter of the textbook, and the section on De Moivre’s Theorem, with additional supplements from several other sources.

This lesson followed a vector unit from chapter 6, *Additional Topics in Trigonometry*, and preceded the lesson on De Moivre's Theorem, also from chapter 6. Graphing complex numbers was not part of the Pre-Calculus A curriculum. Therefore it was assumed that students would have no prior knowledge of the complex plane or graphing complex numbers. It was also a good opportunity to review complex numbers. Lesson 4, Polar Coordinates, from the tenth chapter, *Topics in Analytic Geometry*, directly followed the Complex Plane lesson and provided students with a more detailed rationale for De Moivre's Theorem. Lesson 5 and 6, De Moivre's Theorem came from chapter 6, the same chapter as the unit on vectors.

It is important to note that students had several weeks worth of experience with trigonometry by the time they were exposed to De Moivre's Theorem. These students had been practicing Radian and Degree Measure, Trigonometric Functions and the Unit Circle, Right-Triangle trigonometry, Graphing Trigonometric Functions, Inverse Trigonometric Functions, Using and Verifying Trigonometric Identities, Solving Trigonometric Equations, and working with several formulas such as the Sum and Difference, Multiple-Angle, and Product to Sum formulas. They had also spent several days using the Law of Sines and the Law of Cosines. These were difficult topics for many of these students, more so for those that had not taken the Trigonometric class offered at OCHS. Some students seemed to be growing weary of trigonometry by the time De Moivre's Theorem was taught. This was the culminating trigonometric topic covered in Pre-Calculus B. The next chapter was Conics and students were relieved to find no sign of sine or cosine anywhere in their assignments.

Activity 1 Lesson Plan – Basic Operations of Complex Numbers

Instructor: Cynthia Schneider

Subject: Mathematics

Grade Level: 12^{th} grade

Title: Basic Operations of Complex Numbers

Unit Title: Polynomials and Rational Functions

Content/Topic: This lesson is an introduction to the imaginary unit *i*, and it's use in writing complex numbers in standard form. Students will learn how to add, subtract, and multiply complex numbers, and how to use complex conjugates to divide complex numbers.

Content Objectives: Students will be able to recognize and write complex numbers in standard form. They will be able to perform basic operations of complex numbers, including the use of complex conjugates.

Language Objectives: Students will be able to use the following terminology correctly: imaginary unit *i*, complex numbers, complex conjugate.

Required materials:

- SMART Board lesson including definitions and examples from PRECALCULUS WITH LIMITS: A GRAPHING APPROACH, Third Edition
- Complex Number Operations worksheet #1

Instruction and Practice: See included SMART Board Lesson

Time Allotment: Allow approximately 25 minutes for the lesson. The included assessment will require 30 – 40 minutes depending upon student skill level.

Assessment: See included Complex Number Operations worksheet #1

Activity 1 SMART Board Slides - Operations of Complex Numbers

Slide 1

2.4 Complex Numbers

Use the square root method to solve the quadratic equation.

$$x^2 + 1 = 0$$

Slide 2

Learning Objectives:

1. You will be able to recognize and write complex numbers in standard form.

2. You will be able to perform basic operations of complex numbers

Slide 3

Finding solutions to this type of equation necessitates the use of imaginary numbers.

We define $i = \sqrt{-1}$

Then by adding real parts to multiples of this imaginary unit we obtain the set of **complex numbers**.

Slide 4

If a and b are real numbers, the number $a + bi$ is a complex number, and it is said to be written in standard form.

If $b = 0$ then	$a + bi = a$	Real number
If $b \neq 0$ then	$a + bi$	Imaginary number
If $b \neq 0, a = 0$ then	bi	Pure imaginary number

Slide 5

When we add or subtract imaginary or complex numbers we combine like terms, that is we add or subtract the real parts together and the imaginary parts together.

Sum: $(a + bi) + (c + di) = (a + c) + (b + d)i$

Difference: $(a + bi) - (c + di) = (a - c) + (b - d)i$

Find the sum and difference.

$(4 + 2i) + (3 + 5i) =$

$(6 - 2i) - (4 + 9i) =$

Slide 6

We multiply imaginary numbers using FOIL.

$(a + bi)(c + di) = ac + adi + bci + bdi^2 = (ac - bd) + (ad + bc)i$

First Outside Inside Last

Combine real parts and imaginary parts

$(3 + 2i)(4 + i)$

$(5 + 2i)^2$

$(7 + 3i)(7 - 3i)$

Slide 7

Complex Conjugates:

For $a + bi$ the complex conjugate is $a - bi$

Find the product of the complex conjugates below.

$(a + bi)(a - bi)$

$(1 + i)(1 - i)$

$(2 + 3i)(2 - 3i)$

Slide 8

We use the complex conjugate of the denominator to find the quotient of complex numbers. In this manner we eliminate the imaginary part of the number from the denominator.

$$\frac{8-7i}{1-2i}$$

$$\frac{3+i}{3+3i}$$

$$\frac{7-6i}{i}$$

Slide 9

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Assignment: Operations of Complex Numbers worksheet

Activity 1 - Complex Number Operations

Write the complex number is standard form.

1.) $3+\sqrt{-9}$

2.) $-3i^2+i$

3.) $\left(\sqrt{-75}\right)^2$

Perform the addition or subtraction and write the result in standard form.

4.) $(4+i)+(7-2i)$

5.) $(11-2i)+(-3+6i)$

6.) $\left(7+\sqrt{-18}\right)-\left(3+3i\sqrt{2}\right)$

7.) $13i-(14-7i)$

8.) $22+(-5+8i)+10i$

9.) $-\left(\frac{3}{4}+\frac{7}{5}i\right)-\left(\frac{5}{6}-\frac{1}{6}i\right)$

Perform the multiplication and write the result in standard form.

10.) $\sqrt{-6}\cdot\sqrt{-2}$

11.) $(1+i)(3-2i)$

12.) $(6-2i)(2-3i)$

13.) $\left(\sqrt{14}+i\sqrt{10}\right)\left(\sqrt{14}-i\sqrt{10}\right)$

Find the product of the number and its conjugate.

14.) $4+3i$

15.) $-3+i\sqrt{2}$

Perform the division and write the result in standard form.

16.) $\dfrac{6}{i}$

17.) $\dfrac{4}{4-5i}$

18.) $\dfrac{8-7i}{1-2i}$

19.) $\dfrac{1}{(4-5i)^2}$

20.) $\dfrac{(2-3i)(5i)}{2+3i}$

21.) $\dfrac{2i}{2+i}+\dfrac{5}{2-i}$

Simplify the complex number and write it in standard form.

22.) $4i^2-2i^3$

23.) $-5i^5$

24.) $\left(\sqrt{-2}\right)^6$

Activity 1 – Basic Operations of Complex Numbers (Teacher Version)

Write the complex number is standard form.

1.) $3+\sqrt{-9}$

$= 3+3i$

2.) $-3i^2+i$

$i^2 = -1$

$-3(-1)+i$

$= 3+i$

3.) $\left(\sqrt{-75}\right)^2$

$= \left(5i\sqrt{3}\right)^2$

$= 25i^2 \cdot 3$

$= -75$

Perform the addition or subtraction and write the result in standard form.

4.) $(4+i)+(7-2i)$

$= 11-i$

5.) $(11-2i)+(-3+6i)$

$= 8+4i$

6.) $\left(7+\sqrt{-18}\right)-\left(3+3i\sqrt{2}\right)$

$= \left(7+3i\sqrt{2}\right)-\left(3+3i\sqrt{2}\right)$

$= 4$

7.) $13i-(14-7i)$

$= -14+20i$

8.) $22+(-5+8i)+10i$

$= 17+18i$

9.) $-\left(\frac{3}{4}+\frac{7}{5}i\right)-\left(\frac{5}{6}-\frac{1}{6}i\right)$

$= -\frac{3}{4}-\frac{7}{5}i-\frac{5}{6}+\frac{1}{6}i$

$= \frac{-19}{12}-\frac{37}{30}i$

Perform the multiplication and write the result in standard form.

10.) $\sqrt{-6}\cdot\sqrt{-2}$

$= \sqrt{12}$

$= 2\sqrt{3}$

11.) $(1+i)(3-2i)$

$= 5+i$

12.) $(6-2i)(2-3i)$

$= 6-22i$

13.) $\left(\sqrt{14}+i\sqrt{10}\right)\left(\sqrt{14}-i\sqrt{10}\right)$

$= \left(\sqrt{14}\right)^2-\left(i\sqrt{10}\right)^2$

$= 14-(-10)$

$= 24$

Find the product of the number and its conjugate.

14.) $(4+3i)(4-3i)$

$= 16 - 9i^2$

$= 16 + 9$

$= 25$

15.) $\left(-3+i\sqrt{2}\right)(-3-i\sqrt{2})$

$= 9 - i^2 \cdot 2$

$= 11$

Perform the division and write the result in standard form.

16.) $\frac{6}{i} \cdot \frac{-i}{-i}$

$= \frac{-6i}{-i^2} = -6i$

17.) $\frac{4}{(4-5i)} \frac{(4+5i)}{(4+5i)}$

$= \frac{16+20i}{16-25i^2} = \frac{16+20i}{41}$

18.) $\frac{8-7i}{1-2i} \frac{(1+2i)}{(1+2i)}$

$= \frac{22+9i}{5}$

19.) $\frac{1}{(4-5i)^2}$

$= \frac{1}{(41-40i)} \frac{(41+40i)}{(41+40i)}$

$= \frac{41+40i}{41^2+40^2} = \frac{41+40i}{3281}$

20.) $\frac{(2-3i)(5i)}{2+3i}$

$= \frac{(15+10i)}{(2+3i)} \frac{(2-3i)}{(2-3i)}$

$= \frac{60-25i}{2^2+3^2} = \frac{60-25i}{13}$

21.) $\frac{2i}{2+i} + \frac{5}{2-i}$

$= \frac{2+4i+10+5i}{(2+i)(2-i)}$

$= \frac{12+9i}{4+1} = \frac{12+9i}{5}$

Simplify the complex number and write it in standard form.

22.) $4i^2 - 2i^3$

$= 4(-1) - 2i(-1)$

$= -4 + 2i$

23.) $-5i^5$

$= -5(-1)(-1)i$

$= -5i$

24.) $\left(\sqrt{-2}\right)^6$

$= (i\sqrt{2})^6$

$= 8$

Activity 1 – Basic Operations of Complex Numbers (Student Work)

Write the complex number is standard form.

1.) $3+\sqrt{-9}$

$3+(\sqrt{-1}\sqrt{9})$

$3+(i\,3)$

$\boxed{3+3i}$

2.) $-3i^2+i$

$-3(1)+i$

$\boxed{-3+i}$

3.) $\left(\sqrt{-75}\right)^2$

$\left(\sqrt{i}\sqrt{75}\right)^2$

$i\,75$

$\boxed{75i}$

Perform the addition or subtraction and write the result in standard form.

4.) $(4+i)+(7-2i)$

$\boxed{11-i}$

5.) $(11-2i)+(-3+6i)$

$\boxed{8+4i}$

6.) $\left(7+\sqrt{-18}\right)-\left(3+3i\sqrt{2}\right)$

$\left(7+\sqrt{-1}\sqrt{18}\right)-\left(3+3i\sqrt{2}\right)$

$\left(7+i\,3\sqrt{2}\right)$

$\left(7+3i\sqrt{2}\right)-\left(3+3i\sqrt{2}\right)=\boxed{4}$

7.) $13i-(14-7i)$

$\boxed{-14+20i}$

8.) $22+(-5+8i)+10i$

$17+8i+10i$

$\boxed{17+18i}$

9.) $-\left(\frac{3}{4}+\frac{7}{5}i\right)-\left(\frac{5}{6}-\frac{1}{6}i\right) = -\left(\frac{9}{12}+\frac{42}{30}i\right)-\left(\frac{10}{12}-\frac{5}{30}i\right)$

$\boxed{-\frac{19}{12}-\frac{37}{30}i}$

Perform the multiplication and write the result in standard form.

10.) $\sqrt{-6}\cdot\sqrt{-2}$

$\left(\sqrt{-1}\sqrt{6}\right)\left(\sqrt{-1}\sqrt{2}\right)$

$\left(i\sqrt{6}\right)\left(i\sqrt{2}\right)$

$i\sqrt{12}=\boxed{2i\sqrt{3}}$

11.) $(1+i)(3-2i)$

$3-2i+3i-2i^2$

$3+i-2(1)$

$3+i-2$

$\boxed{1+i}$

12.) $(6-2i)(2-3i)$

$12-18i-4i+6i^2$

$12-22i+6(1)$

$\boxed{18-22i}$

13.) $\left(\sqrt{14}+i\sqrt{10}\right)\left(\sqrt{14}-i\sqrt{10}\right)$

$14-\cancel{i\sqrt{140}}+\cancel{i\sqrt{140}}-10i^2$

$14-10(i^2)$

$14-10$

$\boxed{4}$

Find the product of the number and its conjugate.

14.) $(4+3i)(4-3i)$

$16-\cancel{12i}+\cancel{12i}-9i^2$

$16-9(1)$

$16-9=\boxed{7}\quad(4-3i)$

15.) $\left(-3+i\sqrt{2}\right)\left(-3-i\sqrt{2}\right)$

$9+\cancel{3i\sqrt{2}}-\cancel{3i\sqrt{2}}-2i^2$

$9-2(1)$

$9-2=\boxed{7}\quad(-3-i\sqrt{2})$

Perform the division and write the result in standard form.

16.) $\frac{6}{i}\left(\frac{-i}{-i}\right) = \frac{-6i}{-i^2} = \frac{-6i}{-1} = \boxed{6i}$

17.) $\left(\frac{4}{4-5i}\right)\left(\frac{4+5i}{4+5i}\right) = \frac{16+20i}{16+\cancel{20i}-\cancel{20i}-25i^2}$

$\frac{16+20i}{16-25(1)} = \frac{16+20i}{16-25}$

$\frac{16+20i}{-9}\left(\frac{-1}{-1}\right) = \boxed{\frac{-16-20i}{9}}$

18.) $\left(\frac{8-7i}{1-2i}\right)\left(\frac{1+2i}{1+2i}\right)\ \frac{8+16i-7i-14i^2}{1+\cancel{2i}-\cancel{2i}-4i^2}$

$\frac{8+9i-14}{1-4(1)} = \frac{\overset{2}{-\cancel{6}}+9i}{-\cancel{3}}$

$\boxed{2-3i}$

19.) $\frac{1}{(4-5i)^2}$ $(4-5i)(4-5i)$

$16-20i-20i+25i^2$

$16-40i+25$

$41-40i$

$\frac{1}{(41-40i)}\left(\frac{41+40i}{41+40i}\right)$

$\frac{41+40i}{1681+\cancel{1640i}-\cancel{1640i}-1600i^2} = \frac{41+40i}{1681-1600}$ $\boxed{\frac{41+40i}{81}}$

20.) $\frac{(2-3i)(5i)}{2+3i}$ $10i-15i^2$

$\left(\frac{10i-15}{2+3i}\right)\left(\frac{2-3i}{2-3i}\right)$

$\frac{20i-30i^2-30+45i}{4-\cancel{6i}+\cancel{6i}-9i^2}$

$\frac{20i-60+45i}{4-9}$

$\to \frac{\overset{-13}{\cancel{65}}i-\overset{12}{\cancel{60}}}{-5}$

$-13i+12$

$\boxed{12-13i}$

21.) $\frac{2i}{2+i}+\frac{5}{2-i}$

$\frac{2i(2-i)}{(2+i)(2-i)} + \frac{5(2+i)}{(2-i)(2+i)}$

$\frac{4i-2i^2}{4-\cancel{2i}+\cancel{2i}-i^2} + \frac{10+5i}{4+\cancel{2i}-\cancel{2i}-i^2}$

$\frac{4i-2}{4-1}$ $\frac{10+5i}{4-1}$

$\to \frac{-2+4i}{3}+\frac{10+5i}{3}$

$\frac{-2+4i+10+5i}{3}$

$\boxed{\frac{8+9i}{3}}$

$8/3+3i$

Simplify the complex number and write it in standard form.

22.) $4i^2-2i^3$

$4(1)-2(i)$

$\boxed{4-2i}$

23.) $-5i^5$

$-5(i)$

$\boxed{-5i}$

24.) $\left(\sqrt{-2}\right)^6$

$\left(\sqrt{-1}\sqrt{2}\right)^6$

$\left(i\sqrt{2}\right)^6$

$(i^6 8)$

$\boxed{8}$

Reflection on Activity 1

Students were curious about the origins of complex numbers. I had several students ask why another number system was necessary. A handful of students had already used complex numbers and formed negative opinions regarding their use. I find that students seldom spend time learning how the various sets of numbers, natural integer, rational, irrational, and real numbers are related. Prior to pre-calculus there is little time spent teaching complex numbers let alone that the set of complex numbers contains all other sets of numbers. For most students the actual calculations were easy but the idea of another number system was more then they wanted to think about.

Since this lesson was only introductory, it was process oriented and little time was spent exploring the uses of complex numbers. Students found the process of adding and subtracting very easy. I related this process to combining like terms in an algebraic expression. Once we discussed the strategy for multiplying complex numbers using FOIL, this became an easy calculation. Division was more difficult. About half the class comprehended the concept of a conjugate and cleared complex numbers out of the denominator with ease after a few examples. I used the phrase, "You don't want *i*'s on your bottom" to catch their attention and help them remember to complete this process. It seemed to be effective. I usually relate this to not wanting zeros on "your bottom" or a radical on "your bottom".

The last three problems of the activity worked well spawning several conversations between students regarding the apparent patterns for computing powers of i. I was pleased with this response. I love to see students discussing patterns in mathematics and I enjoy hearing the words and phrases they use to explain math to one another.

Activity 2 Lesson Plan – The Fundamental Theorem of Algebra

Instructor: Cynthia Schneider

Subject: Mathematics

Grade Level: 12th grade

Title: The Fundamental Theorem of Algebra

Unit Title: Polynomials and Rational Functions

Content/Topic: This lesson is an introduction to the Fundamental Theorem of Algebra (FTA). Students will find all the zeros or roots of polynomial functions that lie in the complex number system. Students will rely on factoring, synthetic and long division, and a graphing calculator to locate the zeros or roots.

Content Objectives: Students will be able to use the Fundamental Theorem of Algebra (FTA) to determine the number of zeros of a polynomial function, and then find these zeros or roots, including complex solutions.

Language Objectives: Students will be able to use the following terminology correctly: FTA, linear factorization, roots—zeros—solutions, irreducible.

Required materials:

- SMART Board lesson including definitions and examples from PRECALCULUS WITH LIMITS: A GRAPHING APPROACH, Third Edition
- Fundamental Theorem of Algebra worksheet #2

Instruction and Practice: See included SMART Board Lesson

Time Allotment: Allow approximately 40 minutes for the lesson. The included assessment will require 50 – 60 minutes to complete depending on student skill level.

Assessment: See included Fundament Theorem of Algebra worksheet #2

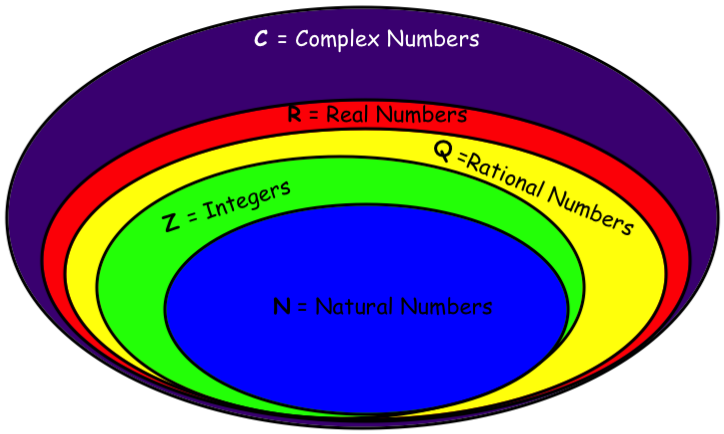
Activity 2 SMART Board Slides – The Fundamental Theorem of Algebra

Slide 1

2.5 The Fundamental Theorem of Algebra

FTA:

If f(x) is a polynomial of degree n, where n > 0, then f has at least one zero in the complex number system.



Slide 2

Learning Objective:

You will be able to use the Fundamental Theorem of Algebra to determine the number of zeros of polynomial function, and find the zeros or roots.

Slide 3

The Linear Factorization Theorem

If f(x) is a polynomial of degree n, where n > 0, then f has precisely n linear factors

$$f(x) = a_n(x - c_1)(x - c_2)\ldots(x - c_n)$$

where $c_1, c_2, \ . \ . \ . \ , c_n$ are complex numbers.

Slide 4

Both of these theorems are referred to as existence theorems.
They don't tell you how to find the zeros or solutions.
For that you rely on:

- Factoring
- Roots
- Quadratic Formula
- Rational Zero Test
- Synthetic or Long Division
- Calculator

Slide 5

EX: Find all the zeros of the function.

$$f(x) = x^4 - 4x^3 + 8x^2 - 16x + 16$$

Slide 6

EX: Find all the zeros of the function.

$$f(x) = x^3 + x^2 + 9x + 9$$

Slide 7

IT IS IMPORTANT TO NOTE THAT COMPLEX SOLUTIONS OF POLYNOMIALS OCCUR IN PAIRS AS COMPLEX CONJUGATES.

$(a + bi)$ and $(a - bi)$

EX: Find a fourth degree polynomial with real coefficients that has -2, -2, and (1 - 4i) as zeros.

Slide 8

Linear and Quadratic Factorization

When the quadratic factors of a function have no rational roots, only irrational or complex roots, the function is said to be irreducible over the rationals.

When the quadratic factors of a function have no real roots, only complex roots (with i) the function is said to be irreducible over the reals.

Slide 9

Irreducible over the rationals
Irreducible over the reals

Write the polynomial as the product of factors that are irreducible over the rationals and as the product of factors irreducible over the reals.

$$f(x) = x^5 - x^4 - x^3 + x^2 - 2x + 2$$

Slide 10

Fundamental Theorem of Algebra worksheet #2

Activity 2 –The Fundamental Theorem of Algebra

Find all the zeros of the function.

1.) $f(x) = x^2(x+3)(x^2-1)$

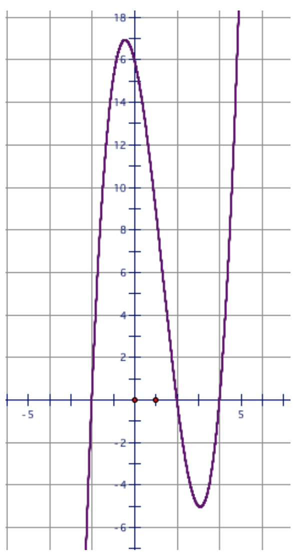
2.) $f(x) = (x+5)(x-8)^2$

3.) $h(t) = (t-3)(t-2)(t-3i)(t+3i)$

4.) $h(m) = (m-4)^2(m-2+4i)(m-2-4i)$

Find all the zeros of the function. Is there a relationship between the number of real zeros and the number of x-intercepts of the graph? Explain.

5.) $f(x) = x^3 - 4x^2 - 4x + 16$



Find all the zeros of the function and write the polynomial as a product of linear factors. Use your graphing calculator to verify your results graphically.

6.) $f(x) = x^3 - 3x^2 - 15x + 125$

7.) $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

Find all the zeros of the function and write the polynomial as a product of linear factors. Use your factorization to determine the x-intercepts of the graph of the function. Use your graphing calculator to verify your results graphically.

8.) $f(x) = x^2 - 12x + 34$

9.) $f(x) = x^3 - 11x + 150$

10.) $f(x) = x^3 + 10x^2 + 33x + 34$

11.) $f(x) = x^4 - 8x^3 + 17x^2 - 8x + 16$

Find a polynomial function with integer coefficients that has the given zeros. (There are many correct answers.)

12.) $4,\ 3i,\ -3i$

13.) $6,\ -5+2i,\ -5-2i$

14.) $-5,\ -5,\ 1+i\sqrt{3}$

Write the polynomial (a) as the product of factors that are irreducible over the *rationals,* (b) as the product of linear and quadratic factors that are irreducible over the *reals,* (c) in completely factored form.

15.) $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$

(*Hint:* One factor is $x^2 - 6$.)

Use the given zero to find all the zeros of the function.

	Function	*Zero*
16.)	$f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$	$2i$
17.)	$h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - i\sqrt{3}$

Graphical Analysis: Use the zero or root feature of a graphing calculator to approximate the zeros of the function accurate to three decimal places. Determine one of the exact zeros and use synthetic division to verify your result. Find the exact values of the remaining zeros.

18.) $f(x) = x^3 + 4x^2 + 14x + 20$

EXPLORATION: Use a graphing calculator to graph the function $f(x) = x^4 - 4x^2 + k$ for different values of *k*. Find values of *k* such that the zeros of *f* satisfy the specified characteristics. (Some parts have many correct answers.)

a.) Four real zeros

b.) Two real zeros each of multiplicity 2

c.) Two real zeros and two complex zeros

d.) Four complex zeros

Activity 2 – The Fundamental Theorem of Algebra (Teacher Version)

Find all the zeros of the function.

1.) $f(x) = x^2(x+3)(x^2-1)$

$0 = x^2(x+3)(x^2-1)$

$x^2 = 0,\ x+3=0,\ x^2-1=0$

$x=0,\ x=-3,\ x=\pm 1$

2.) $f(x) = (x+5)(x-8)^2$

$0 = (x+5)(x-8)^2$

$x+5=0 \quad x-8=0$

$x=-5,\ x=8$

3.) $h(t) = (t-3)(t-2)(t-3i)(t+3i)$

$0 = (t-3)(t-2)(t-3i)(t+3i)$

$t-3=0,\ t-2=0,\ t-3i=0,\ t+3i=0$

$t=3,\ t=2,\ t=3i,\ t=-3i$

4.) $h(m) = (m-4)^2(m-2+4i)(m-2-4i)$

$0 = (m-4)^2(m-2+4i)(m-2-4i)$

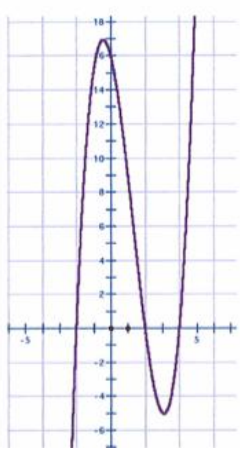
$m-4=0,\ m-2+4i=0,\ m-2-4i=0$

$m=4,\ m=2-4i,\ m=2+4i$

Find all the zeros of the function. Is there a relationship between the number of real zeros and the number of x-intercepts of the graph? Explain.

5.) $f(x) = x^3 - 4x^2 - 4x + 16$

$0 = x^3 - 4x^2 - 4x + 16 \qquad \frac{p}{q} = \pm1, \pm2, \pm4, \pm8, \pm16$



$$\begin{array}{r|rrrr} -2 & 1 & -4 & -4 & 16 \\ & & -2 & 12 & -16 \\ \hline & 1 & -6 & 8 & 0 \end{array}$$

$0 = (x+2)(x^2-6x+8)$

$$\begin{array}{r|rrr} 2 & 1 & -6 & 8 \\ & & 2 & -8 \\ \hline & 1 & -4 & 0 \end{array}$$

$0 = (x+2)(x-2)(x-4)$

$x+2=0,\ x-2=0,\ x-4=0$

$x=-2 \quad x=2 \quad x=4$ real.

Each intercept represents a real solution to the equation, or factor of equation

Find all the zeros of the function and write the polynomial as a product of linear factors. Use your graphing calculator to verify your results graphically.

6.) $f(x) = x^3 - 3x^2 - 15x + 125$

$$\begin{array}{r|rrrr} -5 & 1 & -3 & -15 & 125 \\ & & -5 & 40 & -125 \\ \hline & 1 & -8 & 25 & 0 \end{array}$$

$0 = x^3 - 3x^2 - 15x + 125$

$0 = (x+5)(x^2-8x+25)$

$x = \dfrac{8 \pm \sqrt{64-4(1)(25)}}{2}$

$= 4 \pm 3i$

$0 = (x+5)(x-4+3i)(x-4-3i)$

$x=-5,\ x=4\pm 3i$

7.) $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

$0 = x^4 + 6x^3 + 10x^2 + 6x + 9$

$$\begin{array}{r|rrrrr} -3 & 1 & 6 & 10 & 6 & 9 \\ & & -3 & -9 & -3 & -9 \\ \hline & 1 & 3 & 1 & 3 & 0 \end{array}$$

$0 = (x+3)(x^3+3x^2+x+3)$

$$\begin{array}{r|rrrr} -3 & 1 & 3 & 1 & 3 \\ & & -3 & 0 & -3 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

$0 = (x+3)(x+3)(x^2+1)$

$0 = (x+3)(x+3)(x+i)(x-i)$

$x=-3$

$x=\pm i$

Find all the zeros of the function and write the polynomial as a product of linear factors. Use your factorization to determine the x-intercepts of the graph of the function. Use your graphing calculator to verify your results graphically.

8.) $f(x) = x^2 - 12x + 34$

$x = \dfrac{12 \pm \sqrt{144-4(1)(34)}}{2}$

$= \dfrac{12 \pm 2\sqrt{2}}{2} = 6 \pm \sqrt{2}$

$f(x) = (x - 6 + \sqrt{2})(x - 6 - \sqrt{2})$

x-int: $6+\sqrt{2}$, $6-\sqrt{2}$

Sol: $6 \pm \sqrt{2}$

9.) $f(x) = x^3 - 11x + 150$

$$\begin{array}{r|rrrr} -6 & 1 & 0 & -11 & 150 \\ & & -6 & 36 & -150 \\ \hline & 1 & -6 & 25 & 0 \end{array}$$

$f(x) = (x+6)(x^2 - 6x + 25)$

$x = \dfrac{6 \pm \sqrt{36-4(1)(25)}}{2}$

$x = 3 \pm 3i\sqrt{2}$

$f(x) = (x+6)(x - 3 + 3i\sqrt{2})(x - 3 - 3i\sqrt{2})$

x-int: -6, $3 \pm 3i\sqrt{2}$

Sol: $-6, 3 \pm 3i\sqrt{2}$

10.) $f(x) = x^3 + 10x^2 + 33x + 34$

$$\begin{array}{r|rrrr} -2 & 1 & 10 & 33 & 34 \\ & & -2 & -16 & -34 \\ \hline & 1 & 8 & 17 & 0 \end{array}$$

$x = \dfrac{-8 \pm \sqrt{64-4(1)(17)}}{2} = -4 \pm i$

$f(x) = (x+2)(x^2 + 8x + 17)$

$f(x) = (x+2)(x+4+i)(x+4-i)$

x-int: -2,

Sol $-2, -4 \pm i$

11.) $f(x) = x^4 - 8x^3 + 17x^2 - 8x + 16$

$$\begin{array}{r|rrrrr} 4 & 1 & -8 & 17 & -8 & 16 \\ & & 4 & -16 & 4 & -16 \\ \hline & 1 & -4 & 1 & -4 & 0 \end{array} \qquad \begin{array}{r|rrrr} 4 & 1 & -4 & 1 & -4 \\ & & 4 & 0 & 4 \\ \hline & 1 & 0 & 1 & 0 \end{array}$$

$f(x) = (x-4)(x-4)(x^2+1)$

$f(x) = (x-4)(x-4)(x+i)(x-i)$

x-int: 4, sol: $x = 4$, $x = \pm i$

Find a polynomial function with integer coefficients that has the given zeros. (There are many correct answers.)

12.) 4, $3i$, $-3i$ $f(x) = (x-4)(x-3i)(x+3i) = x^3 - 4x^2 + 9x - 36$

13.) 6, $-5+2i$, $-5-2i$ $f(x) = (x-6)(x+5+2i)(x+5-2i) = x^3 + 4x^2 - 31x - 174$

14.) -5, -5, $1+i\sqrt{3}$ $f(x) = (x+5)(x+5)(x-1+i\sqrt{3})(x-1-i\sqrt{3})$

$= x^4 + 8x^3 + 9x^2 - 10x + 100$

Write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, (c) in completely factored form.

15.) $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$

(*Hint:* One factor is $x^2 - 6$.)

$$\begin{array}{r l} & x^2 - 2x + 3 \\ x^2 + 0x - 6 \,\big) & x^4 - 2x^3 - 3x^2 + 12x - 18 \\ & -(x^4 + 0x^3 - 6x^2) \\ \hline & -2x^3 + 3x^2 + 12x \\ & -(-2x^3 + 0x^2 + 12x) \\ \hline & 3x^2 + 0x - 18 \\ & -(3x^2 + 0x - 18) \\ \hline & 0 \end{array}$$

$x = \dfrac{2 \pm \sqrt{4-4(1)(3)}}{2} = 1 \pm i\sqrt{2}$

a.) $f(x) = (x^2 - 6)(x^2 - 2x + 3)$ irreducible over rationals

b.) $f(x) = (x + \sqrt{6})(x - \sqrt{6})(x^2 - 2x + 3)$ irreducible over reals

c.) $f(x) = (x + \sqrt{6})(x - \sqrt{6})(x - 1 + i\sqrt{2})(x - 1 - i\sqrt{2})$

$(x-2i)(x+2i)$ **Use the given zero to find all the zeros of the function.**

$= x^2+4$

Function	*Zero*
16.) $f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$	$2i$, $-2i$, 1, $-\frac{1}{2}$
17.) $h(x) = 3x^3 - 4x^2 + 8x + 8$	$1 - i\sqrt{3}$, $1+i\sqrt{3}$, $-\frac{2}{3}$

$$\begin{array}{r} 2x^2 - x - 1 \\ x^2+0x+4 \,\overline{)\, 2x^4 - x^3 + 7x^2 - 4x - 4} \\ -(2x^4+0x^3+8x^2) \\ \hline -x^3 - x^2 - 4x \\ -(-x^3+0x^2-4x) \\ \hline -x^2 - 0x - 4 \end{array}$$

$f(x) = (x-2i)(x+2i)(2x^2-x-1)$

$x = \frac{1 \pm \sqrt{1^2 - (4)(2)(-1)}}{4}$

$= \frac{1 \pm 3}{4}$ $x = 1$ or $x = -\frac{1}{2}$

$(x-1-i\sqrt{3})(x-1+i\sqrt{3})$

$= (x-1)^2 - 3i^2$

$= x^2 - 2x + 1 + 3$

$= x^2 - 2x + 4$

$$\begin{array}{r} 3x - 2 \\ x^2-2x+4 \,\overline{)\, 3x^3 - 4x^2 + 8x + 8} \\ -(3x^3-6x^2+12x) \\ \hline 2x^2 - 4x + 8 \\ -(2x^2-4x+8) \end{array}$$

$3x - 2 = 0$

$3x = 2$

$x = \frac{2}{3}$

$h(x) = (x-1-i\sqrt{3})(x-1+i\sqrt{3})(3x-2)$

Graphical Analysis: Use the zero or root feature of a graphing calculator to approximate the zeros of the function accurate to three decimal places. Determine one of the exact zeros and use synthetic division to verify your result. Find the exact values of the remaining zeros.

root $x \approx -2$

18.) $f(x) = x^3 + 4x^2 + 14x + 20$

EXACT zeros:

$x = -2$

$x = -1 \pm 3i$

$$\begin{array}{r|rrrr} -2 & 1 & 4 & 14 & 20 \\ & & -2 & -4 & -20 \\ \hline & 1 & 2 & 10 & 0 \end{array}$$

$f(x) = (x+2)(x^2+2x+10)$

$x = \frac{-2 \pm \sqrt{4 - 4(1)(10)}}{2}$

$= \frac{-2 \pm 6i}{2}$

$= -1 \pm 3i$

EXPLORATION: Use a graphing calculator to graph the function $f(x) = x^4 - 4x^2 + k$ for different values of *k*. Find values of *k* such that the zeros of *f* satisfy the specified characteristics. (Some parts have many correct answers.)

a.) Four real zeros EX $k=1$ $x \approx \pm 1.932$, $x \approx \pm 0.518$

b.) Two real zeros each of multiplicity 2 $k=4$ $(x-2)(x-2)(x+2)(x+2)$

c.) Two real zeros and two complex zeros $k=-6$ $x = \pm 2$

d.) Four complex zeros $k=6$

Activity 2 – The Fundamental Theorem of Algebra (Student Work)

Find all the zeros of the function.

1.) $f(x) = x^2(x+3)(x^2-1)$

$x = 0, -3, \pm 1$

2.) $f(x) = (x+5)(x-8)^2$

$x = -5, 8, 8$

3.) $h(t) = (t-3)(t-2)(t-3i)(t+3i)$

$x = 3, 2, 3i, -3i$

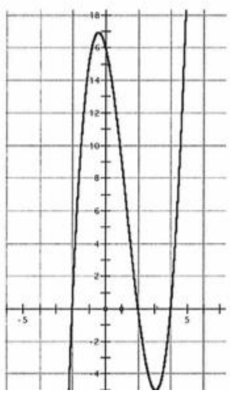
4.) $h(m) = (m-4)^2(m-2+4i)(m-2-4i)$

$x = 4, 4, 2 \pm 4i$

$0 = m - 2 - 4i$ $\quad 0 = m - 2 + 4i$
$2 = m - 4i$ $\quad 2 = m + 4i$
$2 + 4i = m$ $\quad m = 2 - 4i$
$m = 2 + 4i$

Find all the zeros of the function. Is there a relationship between the number of real zeros and the number of x-intercepts of the graph? Explain.

5.) $f(x) = x^3 - 4x^2 - 4x + 16$



$f(x) = x^3 - 4x^2 - 4x + 16$

$x = -2, 2, 4$

There are three real zeros that are solutions and three x-intercepts on the graph.

Find all the zeros of the function and write the polynomial as a product of linear factors. Use your graphing calculator to verify your results graphically.

6.) $f(x) = x^3 - 3x^2 - 15x + 125$ $\quad (x+5)$

$x = -5, 4 \pm 3i$

$$\begin{array}{r|rrrr} -5 & 1 & -3 & -15 & +125 \\ & & -5 & 40 & -125 \\ \hline & 1 & -8 & 25 & 0 \end{array}$$

$x^2 - 8x + 25$

$\boxed{(x+5)(x-4+3i)(x-4-3i)}$

$\dfrac{8 \pm \sqrt{64 - 4(25)}}{2}$

$\dfrac{8 \pm \sqrt{-36}}{2}$ $\quad \dfrac{8 \pm 6i}{2} = 4 \pm 3i$

7.) $h(x) = x^4 + 6x^3 + 10x^2 + 6x + 9$

$x = -3, -3, \pm i$ $\quad \boxed{(x+3)(x+3)(x+i)(x-i)}$

$$\begin{array}{r|rrrrr} -3 & 1 & 6 & 10 & 6 & 9 \\ & & -3 & -9 & -3 & -9 \\ \hline & 1 & 3 & 1 & 3 & 0 \end{array}$$

$x^3 + 3x^2 + x + 3$ $\quad (x+3)$

$(x^2 + 1)(x + 3)$ $\quad x^2 + 1 = 0$

$x^3 + 3x^2 + x - 3$ $\quad x^2 = -1$

$x = \sqrt{-1} = i$

Find all the zeros of the function and write the polynomial as a product of linear factors. Use your factorization to determine the x-intercepts of the graph of the function. Use your graphing calculator to verify your results graphically.

8.) $f(x) = x^2 - 12x + 34$

$\frac{-b \pm \sqrt{b^2-4ac}}{2a}$

$\frac{12 \pm \sqrt{144-4(34)}}{2}$

$\frac{12 \pm \sqrt{8}}{2}$

$\frac{12 \pm 2\sqrt{2}}{2}$

$\boxed{x = 6 \pm \sqrt{2}}$

$\boxed{(x-6+\sqrt{2})(x-6-\sqrt{2})}$

9.) $f(x) = x^3 - 11x + 150$

$\boxed{x = -6,\ 3 \pm 4i}$

$\boxed{(x+6)(x-3+4i)(x-3-4i)}$

$$\begin{array}{r|rrrr} -6 & 1 & 0 & -11 & 150 \\ & & -6 & 36 & -150 \\ \hline & 1 & -6 & 25 & 0 \end{array}$$

$x^2 - 6x + 25$

$\frac{6 \pm \sqrt{36-4(25)}}{2}$

$\frac{6 \pm \sqrt{-64}}{2}$

$\frac{6 \pm 8i}{2} = 3 \pm 4i$

10.) $f(x) = x^3 + 10x^2 + 33x + 34$

$\boxed{(x+2)(x+4+i)(x+4-i)}$

$\boxed{x = -4 \pm i, -2}$

$$\begin{array}{r|rrrr} -2 & 1 & 10 & 33 & 34 \\ & & -2 & -16 & -34 \\ \hline & 1 & 8 & 17 & 0 \end{array}$$

$x^2 + 8x + 17$

$\frac{-8 \pm \sqrt{64-4(17)}}{2}$

$\frac{-8 \pm \sqrt{-4}}{2}$

$\frac{-8 \pm 2i}{2} = -4 \pm i$

11.) $f(x) = x^4 - 8x^3 + 17x^2 - 8x + 16$

$\boxed{(x-4)(x-4)(x+i)(x-i)}$

$\boxed{x = 4, 4, \pm i}$

$$\begin{array}{r|rrrrr} 4 & 1 & -8 & 17 & -8 & 16 \\ & & 4 & -16 & 4 & -16 \\ \hline & 1 & -4 & 1 & -4 & 0 \end{array}$$

$x^3 - 4x^2 + x - 4$

$(x^2+1)(x-4)$

$x^2 = -1$

$x = \sqrt{-1}$

$x = \pm i$

Find a polynomial function with integer coefficients that has the given zeros. (There are many correct answers.)

12.) 4, $3i$, $-3i$

$(x-4)(x-3i)(x+3i) = (x^2 - 3ix - 4x + 12i)(x+3i) = x^3 + x^2 3i - x^2 3i - 9i^2 x - 4x^2 - 12xi + 12xi + 36i^2$

$x^3 \cancel{+x^2 3i - x^2 3i} - 9x - 4x^2 \cancel{-12xi + 12xi} + 36 = \boxed{x^3 - 4x^2 - 9x + 36}$

13.) 6, $-5+2i$, $-5-2i$

$(x-6)(x+5+2i)(x+5-2i) = (x^2 + 5x + 2xi - 6x - 30 - 12i)(x+5-2i) =$

$\boxed{x^3 + 4x^2 - 31x - 174}$

14.) -5, -5, $1+i\sqrt{3}$

$(x+5)(x+5)(x-1+i\sqrt{3})(x-1-i\sqrt{3}) =$

$(x^2 + 5x + 5x + 25)((x-1) - i\sqrt{3})((x-1) + i\sqrt{3}) = (x^2 + 10x + 25)((x-1)^2 + \ldots)$

$\boxed{x^2 + 10x + 25}\ ((x-1)(x-1) - 3) \quad (x^2 + 10x + 25)(x^2 - 2x + 2) =$

$-50x - 50$

$\boxed{x^4 + 8x^3 + 3x^2 - 70x - 50}$

Write the polynomial (a) as the product of factors that are irreducible over the *rationals*, (b) as the product of linear and quadratic factors that are irreducible over the *reals*, (c) in completely factored form.

15.) $f(x) = x^4 - 2x^3 - 3x^2 + 12x - 18$

(*Hint:* One factor is $x^2 - 6$.)

a) $\boxed{f(x) = (x^2 - 6)(x^2 - 2x + 3)}$ irreducible over the rationals

b) $\boxed{f(x) = (x - \sqrt{6})(x + \sqrt{6})(x^2 - 2x + 3)}$ irreducible over reals

c) $\boxed{f(x) = (x - \sqrt{6})(x + \sqrt{6})(x - 1 + i\sqrt{2})(x - 1 - i\sqrt{2})}$ completely factored

$$\begin{array}{r|l} & x^2 - 2x + 3 \\ \hline x^2 + 0x - 6 & x^4 - 2x^3 - 3x^2 + 12x - 18 \\ & -(x^4 + 0x^3 - 6x^2) \\ \hline & -2x^3 + 3x^2 + 12x \\ & -(-2x^3 + 0x^2 + 12x) \\ \hline & 3x^2 + 0x - 18 \\ & -(3x^2 + 0x - 18) \\ \hline & 0 \end{array}$$

$\frac{2 \pm \sqrt{4 - 4(3)}}{2} = \frac{2 \pm \sqrt{-8}}{2} = \frac{2 \pm 2i\sqrt{2}}{2}$

$1 \pm i\sqrt{2}$

Use the given zero to find all the zeros of the function.

Function *Zero*

16.) $f(x) = 2x^4 - x^3 + 7x^2 - 4x - 4$ $\boxed{x = 1, -0.5, \pm 2i}$ $\pm 2i$

$$\begin{array}{r|rrrrr} 1 & 2 & -1 & 7 & -4 & -4 \\ & & 2 & 1 & 8 & 4 \\ \hline -.5 & 2 & 1 & 8 & 4 & \emptyset \\ & & -1 & 0 & -4 & \\ \hline & 2 & 0 & 8 & \emptyset & \end{array}$$

$f(x) = (x-1)(x+0.5)(x+2i)(x-2i)$

$2x^2 + 8 = 0$ $2x^2 = -8$ $\sqrt{x^2} = \sqrt{-4}$ $x = \pm\sqrt{-4} =$ $x = \pm 2i$

17.) $h(x) = 3x^3 - 4x^2 + 8x + 8$ $1 - i\sqrt{3}$ $\boxed{x = -2/3, 1 \pm i\sqrt{3}}$

$((x-1) - i\sqrt{3})((x-1) + i\sqrt{3})$

$(x-1)^2 + \cancel{(x-1)(i\sqrt{3}) - (x-1)(i\sqrt{3})} - 3i^2$ (-1)

$(x-1)^2 + 3$

$(x-1)(x-1) + 3 = x^2 - 2x + 1 + 3 = x^2 - 2x + 4$

$$\begin{array}{r|l} & 3x + 2 \\ \hline x^2 - 2x + 4 & 3x^3 - 4x^2 + 8x + 8 \\ & -(3x^3 - 6x^2 + 12x) \\ \hline & 2x^2 - 4x + 8 \\ & -(2x^2 - 4x + 8) \\ \hline & \emptyset \end{array}$$

$3x + 2 = 0$

$3x = -2$

$x = -2/3$

Graphical Analysis: Use the zero or root feature of a graphing calculator to approximate the zeros of the function accurate to three decimal places. Determine one of the exact zeros and use synthetic division to verify your result. Find the exact values of the remaining zeros.

18.) $f(x) = x^3 + 4x^2 + 14x + 20$ $x = -2.000$

$$\begin{array}{r|rrrr} -2 & 1 & 4 & 14 & 20 \\ & & -2 & -4 & -20 \\ \hline & 1 & 2 & 10 & \emptyset \end{array}$$

$f(x) = (x+2)(x+1+3i)(x+1-3i)$

$\boxed{x = -2, -1 \pm 3i}$

$x^2 + 2x + 10$

$\dfrac{-2 \pm \sqrt{4 - 4(10)}}{2} = \dfrac{-2 \pm \sqrt{-36}}{2} = \dfrac{-2 \pm (\sqrt{-1} \cdot 6)}{2}$

$= \dfrac{-\cancel{2} \pm \cancel{6}^{3} i}{\cancel{2}} = -1 \pm 3i$

EXPLORATION: Use a graphing calculator to graph the function $f(x) = x^4 - 4x^2 + k$ for different values of *k*. Find values of *k* such that the zeros of *f* satisfy the specified characteristics. (Some parts have many correct answers.)

a.) Four real zeros $\boxed{K = 1}$ $x = -1.9, -.52, .52, 1.9$

b.) Two real zeros each of multiplicity 2 $\boxed{K = 4}$ $x = -1.4, -1.4, 1.4, 1.4$

c.) Two real zeros and two complex zeros $\boxed{K = -6}$ $x = -2.3, 2.3$

d.) Four complex zeros $\boxed{K = 8}$

Reflection on Activity 2

The second SMART Board was useful since it provided students with a concepts map to help them think through the various sets of numbers. I noticed that many students drew the ven diagram and referred back to it later. The Fundamental Theorem of Algebra seemed insignificant to most students but they frequently made mention of the Linear Factorization Theorem when talking to one another. They reminded each other that there were “n solutions” to an equation they were working with. When finding solutions of polynomials students were most likely to rely on their calculators. This of course was the fastest way to find solutions but I had hoped some would remember to look at the possible set of rational solutions, that is

$\frac{p}{q} = \frac{\textbf{\textit{factors of the contant term}}}{\textbf{\textit{factors of the leading coefficient}}}$, and use synthetic division to confirm. I did require students to verify solutions from the calculator with synthetic division.

Several students struggled with linear factorization and many forgot to include the <u>pair</u> of complex conjugates. They then wondered why their neighbors were getting results with a higher leading exponent, or they wondered what to do with the *i* left over when they multiplied terms back together. This became a quick way for me to check understanding.

Most students found the terminology “irreducible over the rationals” or “irreducible over the reals” very confusing. With continued practice the factoring, and writing linear factorizations became easier. However for several students it was a formulaic process with specific clues at to when to write irreducible over the rationals or reals without understanding why.

Problem number 15 was a good opportunity to review long division and it provided students with an example of a polynomial that was factorable but had no rational solutions.

The last problem gave students a chance to examine how the position of a graph is altered by the constant term and reinforced the concepts of zeros as solutions and set the stage for the graphical representation of complex solutions.

Activity 3 Lesson Plan – Complex Numbers and the Complex Plane

Instructor: Cynthia Schneider

Subject: Mathematics

Grade Level: 12^{th} grade

Title: Complex Numbers and The Complex Plane

Unit Title: Additional Topics in Trigonometry

Content/Topic: This lesson is an introduction to the complex plane. Students will learn how to plot complex numbers given and real axis and an imaginary axis. They will also learn to find the absolute value of a complex number.

Content Objectives: Students will be able to plot complex numbers on the complex plane and find the absolute value of a complex number.

Language Objectives: Students will be able to understand and use the following terminology: complex plane, real axis, imaginary axis, and absolute value of a complex number.

Required materials:

- SMART Board lesson including definitions and examples from PRECALCULUS WITH LIMITS: A GRAPHING APPROACH, Third Edition
- Graphing Complex Numbers worksheet #3

Instruction and Practice: See included SMART Board Lesson

Time Allotment: Allow approximately 20 minutes for the lesson. The included assessment will require 20 – 25 minutes to complete depending on student skill level.

Assessment: See included Graphing Complex Numbers worksheet #3

Activity 3 SMART Board Slides – Complex Numbers and the Complex Plane

Slide 1

Graphing Complex Numbers on the Complex Plane

What you should learn:

How to plot complex numbers on the complex plane

How to find the absolute value of a complex number

Slide 2

Remember:

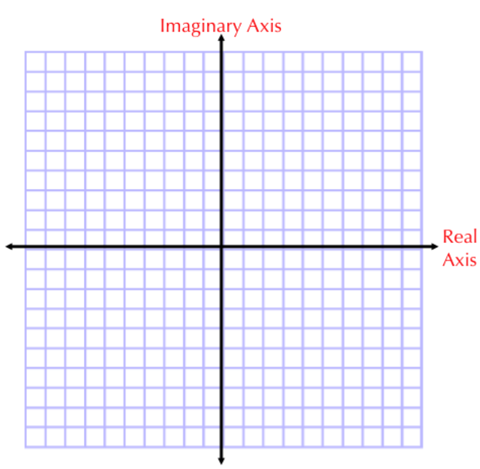
Complex Numbers take the form $a + bi$ where a is the real part and b is the imaginary part.

Also $i = \sqrt{-1}$.

Slide 3

The Complex Plane

$a + bi \longrightarrow (a, b)$



$(2 + 3i)$ $(-4 - 2i)$ $(9 - i)$

Slide 4

The Absolute Value of a Complex Number

The absolute value of the complex number z = a + b*i* is

$$|a + bi| = \sqrt{a^2 + b^2}$$

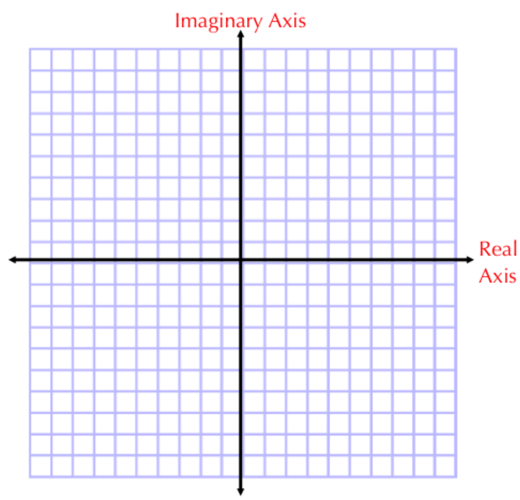
The absolute value of a complex number a + b*i* is defined as the distance between the origin (0, 0) and the point (a, b).

Find the absolute value of the complex number.

(2 + 3*i*)　　　(-4 - 2*i*)　　　(9 - *i*)

Slide 5

EX: Plot the complex number (6 + 8*i*).
Find the absolute value of the complex number.



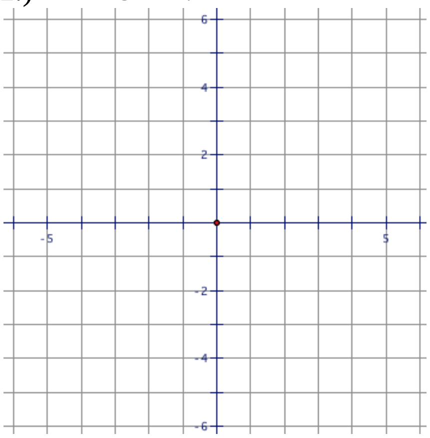
Slide 6

☆ Homework: Complex Graphing Worksheet

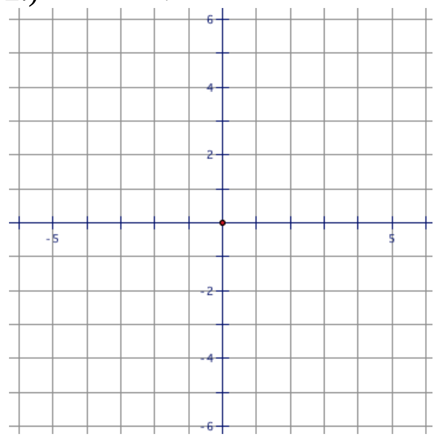
Activity 3 – Complex Numbers and the Complex Plane

Express each complex number as an ordered pair and then graph each number on the complex plane.

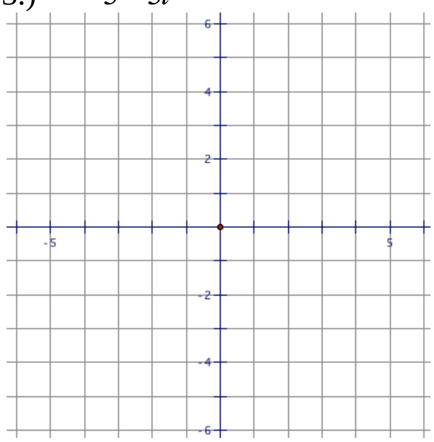
1.) $-3-4i$



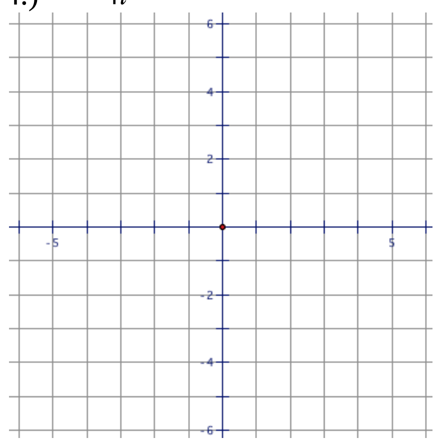
2.) $-4+i$



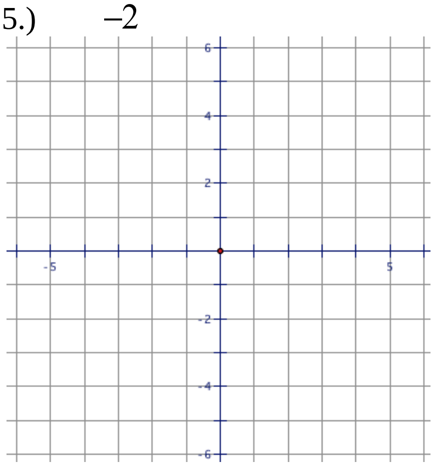
3.) $3-3i$



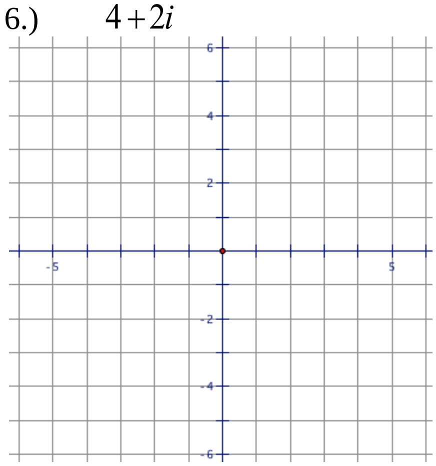
4.) $4i$



5.) -2

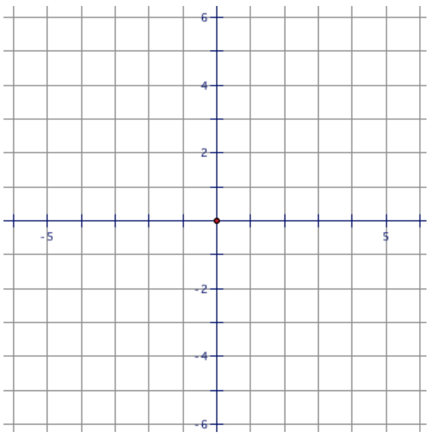


6.) $4+2i$

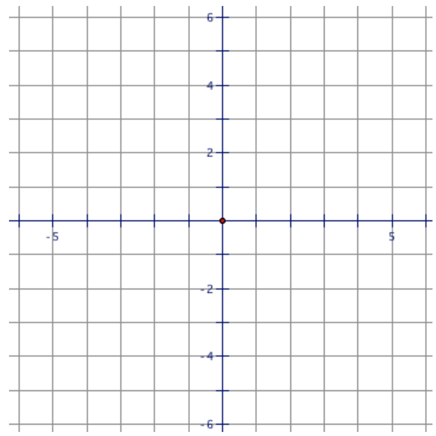


Calculate the absolute value of each number and then graph each number on the complex plane.

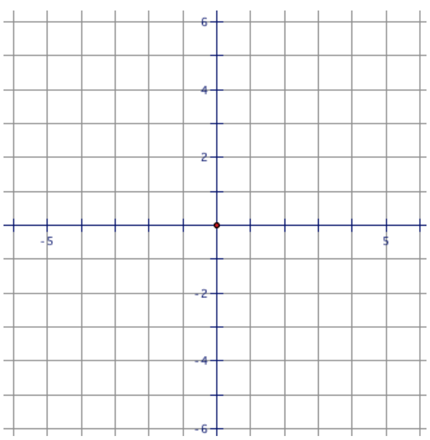
7.) $-1-i$



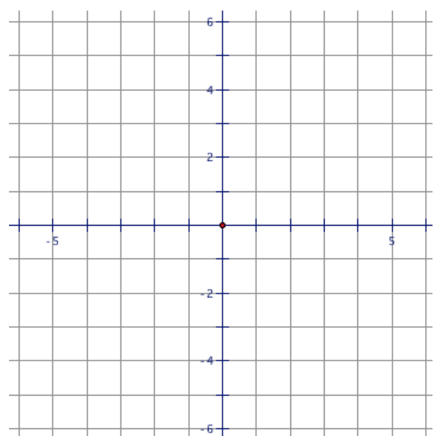
8.) $-1+i$



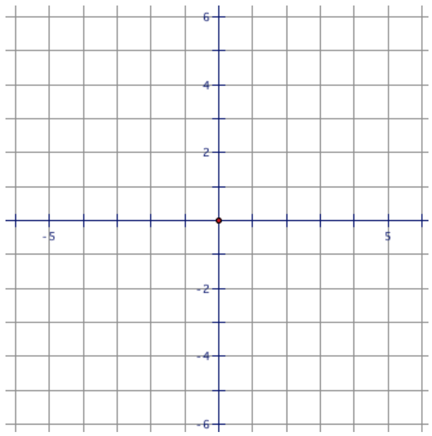
9.) $-2+2i$



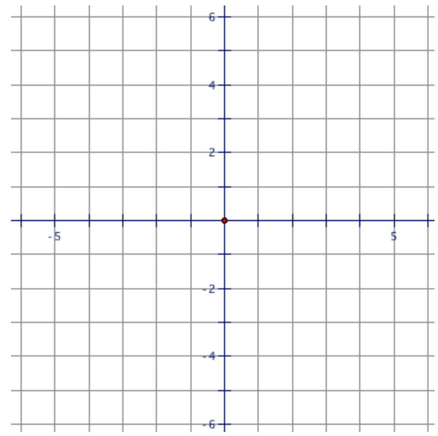
10.) $2-2i$



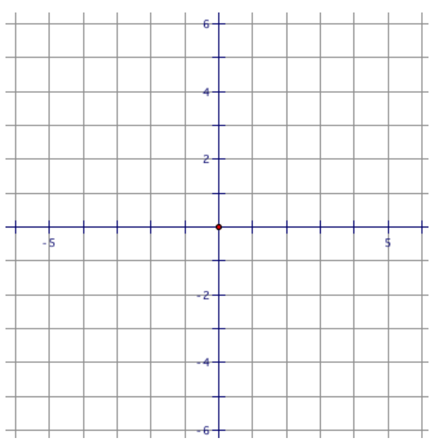
11.) $\sqrt{3} - i\sqrt{3}$



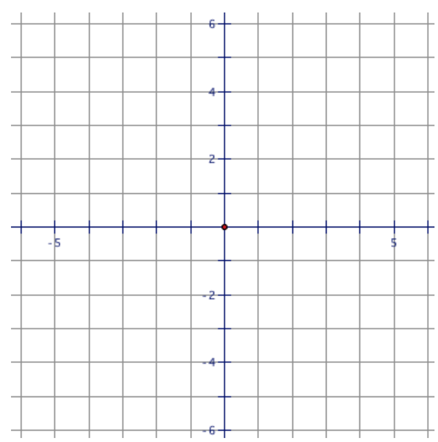
12.) $-\sqrt{5} + i\sqrt{5}$



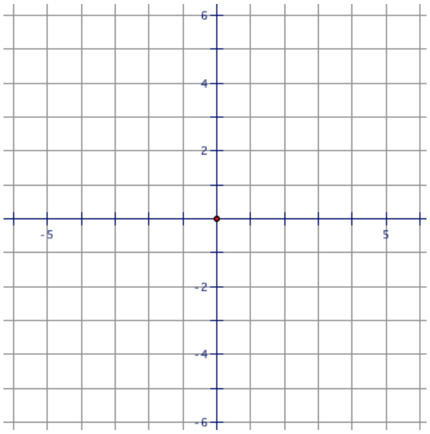
13.) $-3i$



14.) $\sqrt{2} - 2i\sqrt{2}$

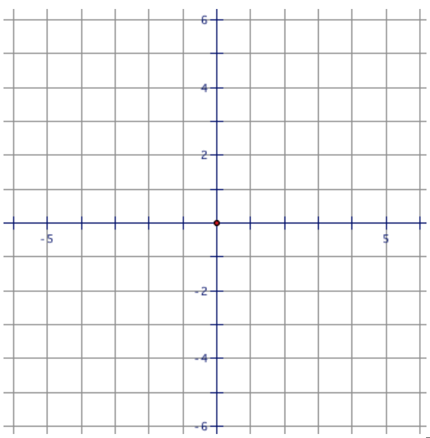


15.) $-5+5i\sqrt{5}$

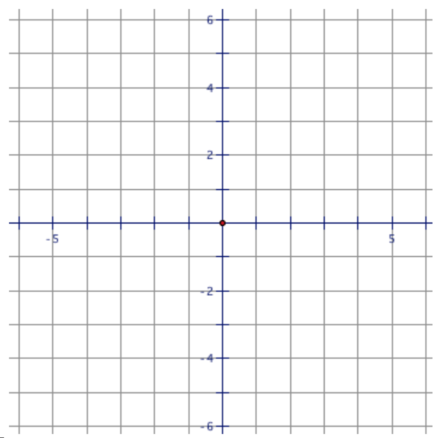


Find each sum graphically. Check algebraically.

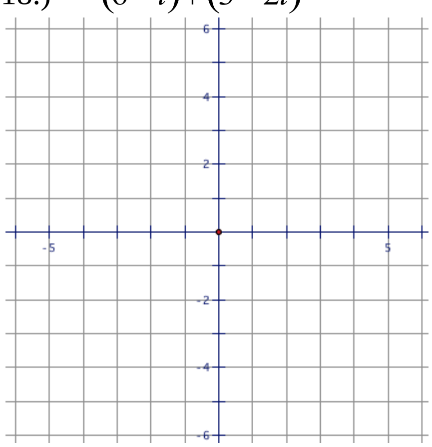
16.) $(4+i)+(-4+5i)$



17.) $(3+2i)+(-2+4i)$



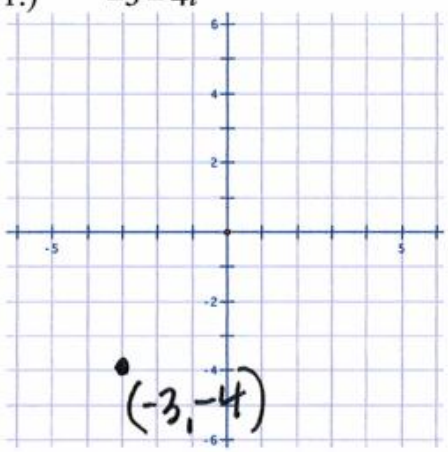
18.) $(6-i)+(3-2i)$



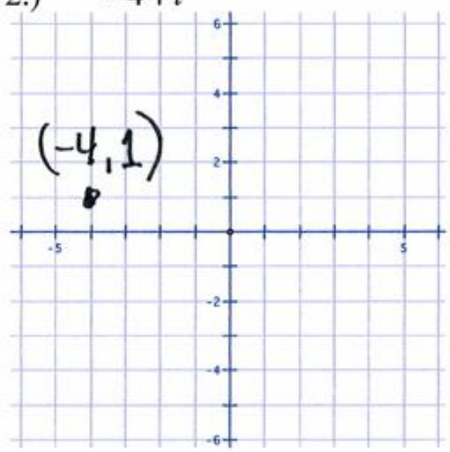
Activity 3 – Complex Numbers on the Complex Plane (Teacher's Version)

Express each complex number as an ordered pair and then graph each number on the complex plane.

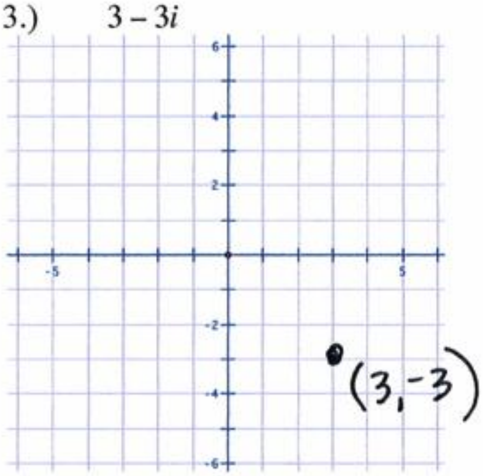
1.) $-3-4i$



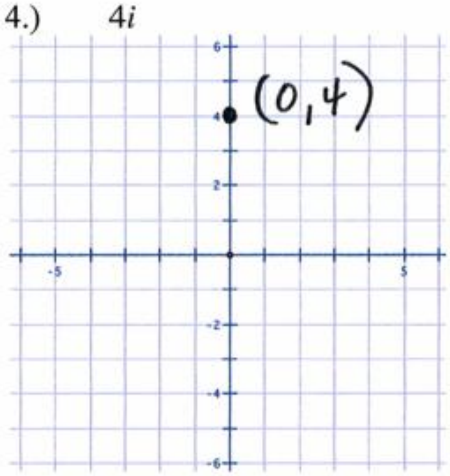
2.) $-4+i$



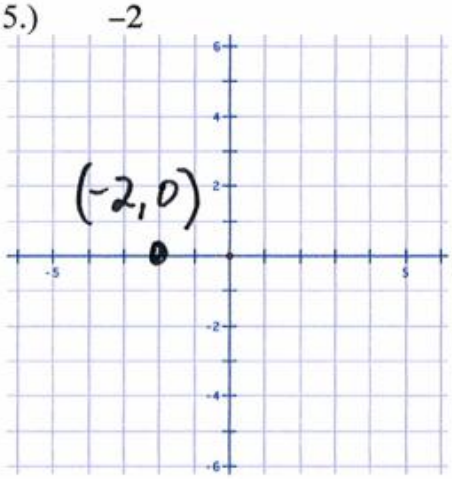
3.) $3-3i$



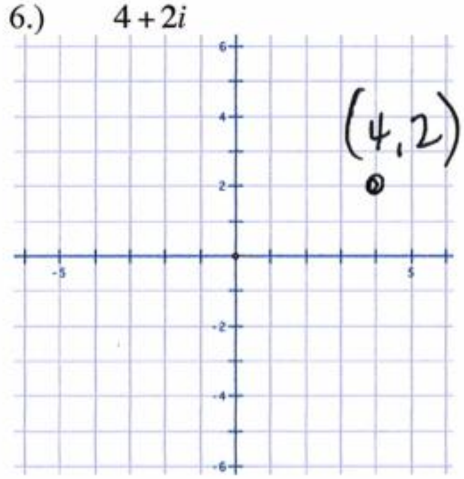
4.) $4i$



5.) -2



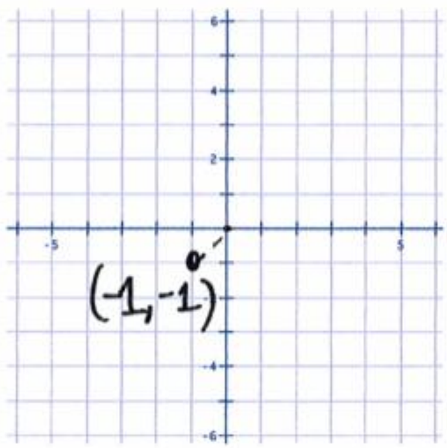
6.) $4+2i$



Calculate the absolute value of each number and then graph each number on the complex plane.

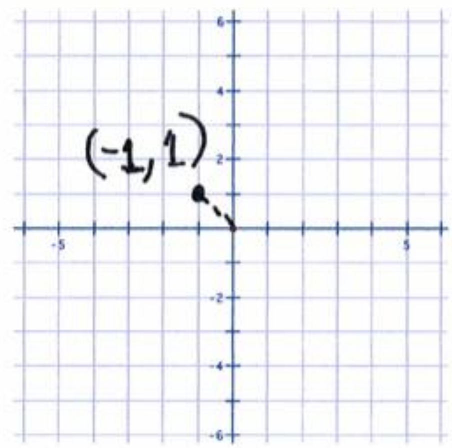
7.) $-1-i$

$|-1-i| = \sqrt{(-1)^2+(-1)^2} = \sqrt{2}$



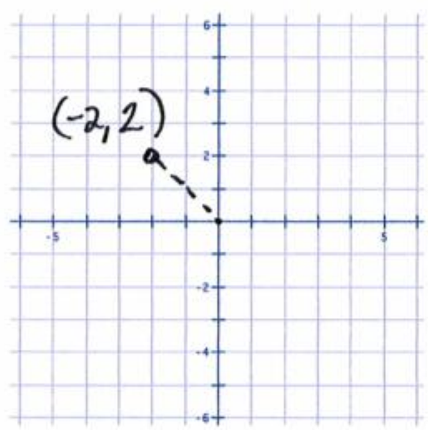
8.) $-1+i$

$|-1+i| = \sqrt{(-1)^2+1^2} = \sqrt{2}$



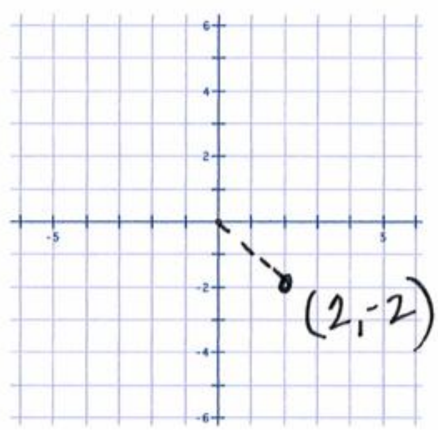
9.) $-2+2i$

$|-2+2i| = \sqrt{(-2)^2+2^2} = \sqrt{8} = 2\sqrt{2}$



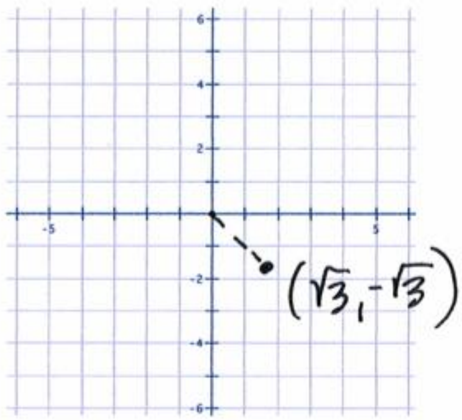
10.) $2-2i$

$|2-2i| = \sqrt{2^2+(-2)^2} = \sqrt{8} = 2\sqrt{2}$



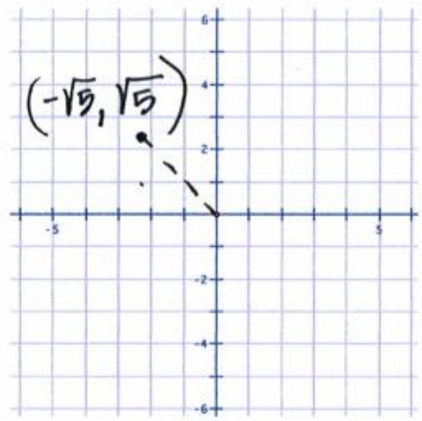
11.) $\sqrt{3} - i\sqrt{3}$

$$\left|\sqrt{3} - i\sqrt{3}\right| = \sqrt{(\sqrt{3})^2 + (-\sqrt{3})^2} = \sqrt{6}$$



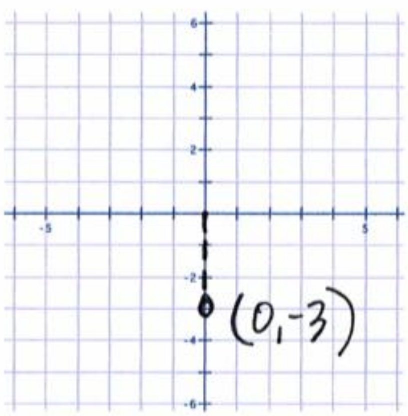
12.) $-\sqrt{5} + i\sqrt{5}$

$$\left|-\sqrt{5} + i\sqrt{5}\right| = \sqrt{(-\sqrt{5})^2 + (\sqrt{5})^2} = \sqrt{10}$$



13.) $-3i$

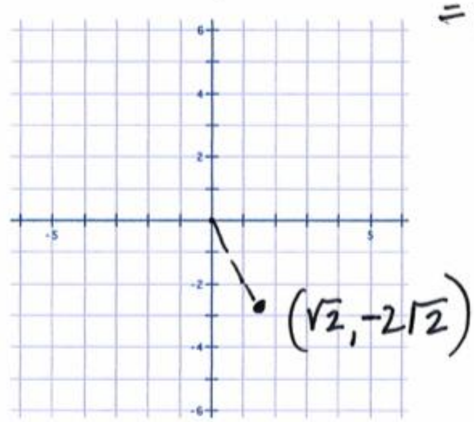
$$\left|-3i\right| = \sqrt{0^2 + (-3)^2} = 3$$



14.) $\sqrt{2} - 2i\sqrt{2}$

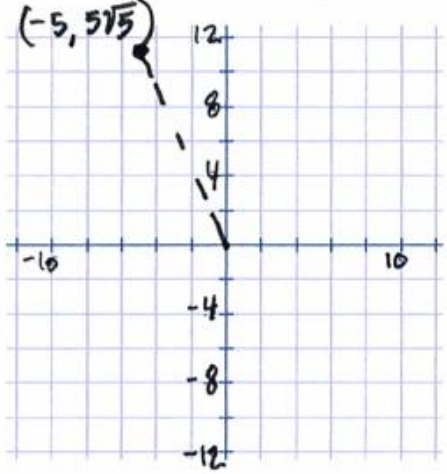
$$\left|\sqrt{2} - 2i\sqrt{2}\right| = \sqrt{(\sqrt{2})^2 + (-2\sqrt{2})^2} = \sqrt{2+8}$$

$$= \sqrt{10}$$



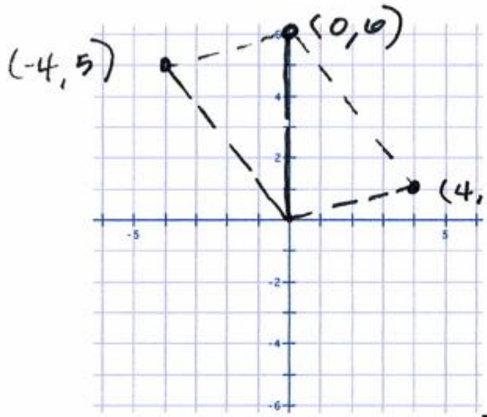
15.) $-5+5i\sqrt{5}$ $\quad |-5+5i\sqrt{5}| = \sqrt{(-5)^2+(5\sqrt{5})^2}$

$$= \sqrt{25+125}$$
$$= \sqrt{150}$$
$$= 5\sqrt{6}$$

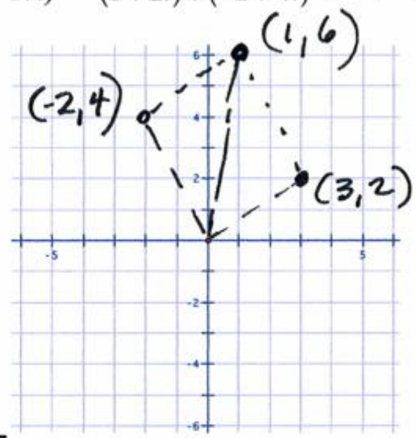


Find each sum graphically. Check algebraically.

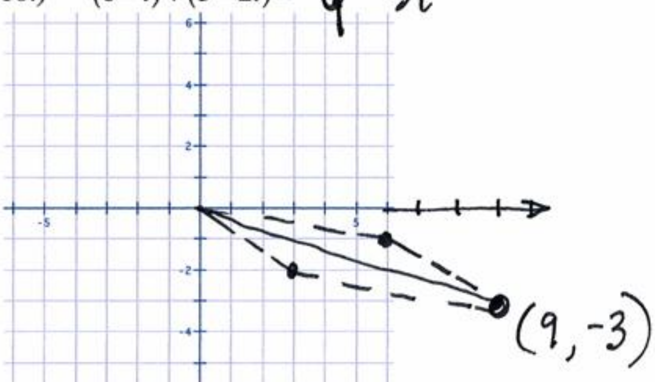
16.) $(4+i)+(-4+5i) = 0+6i$



17.) $(3+2i)+(-2+4i) = 1+6i$



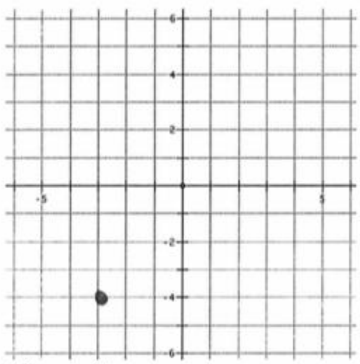
18.) $(6-i)+(3-2i) = 9-3i$



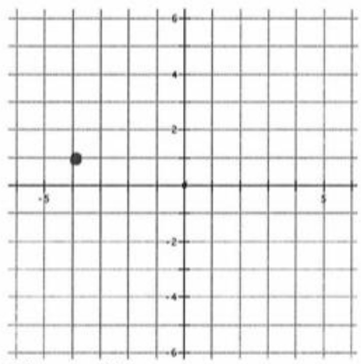
Activity 3 – Complex Numbers on the Complex Plane (Student Work)

Express each complex number as an ordered pair and then graph each number on the complex plane.

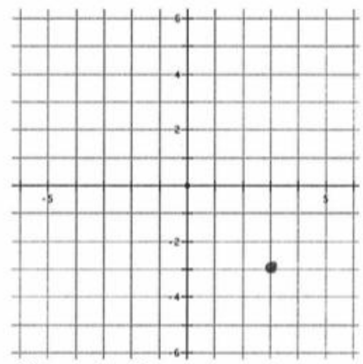
1.) $-3-4i$ (-3, -4i)



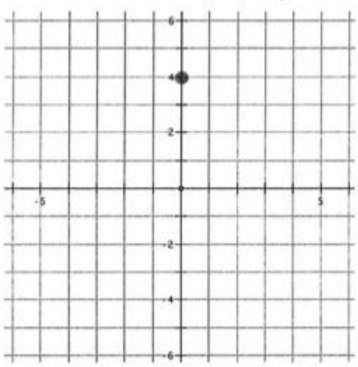
2.) $-4+i$ (-4, i)



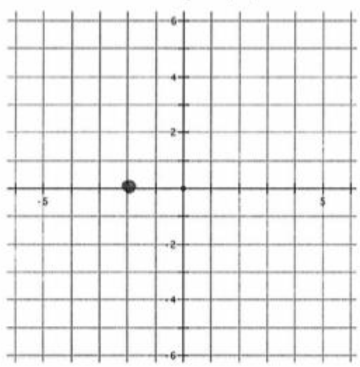
3.) $3-3i$ (3, -3i)



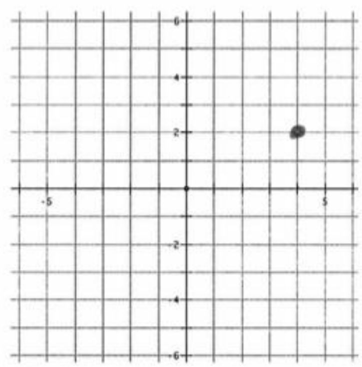
4.) $4i$ (0, 4i)



5.) -2 (-2, 0)

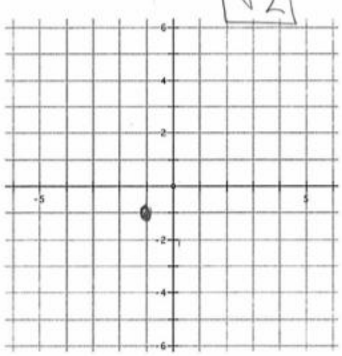


6.) $4+2i$ (4, 2i)

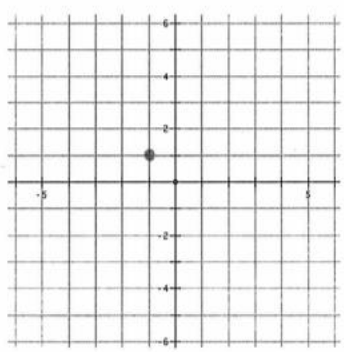


Calculate the absolute value of each number and then graph each number on the complex plane.

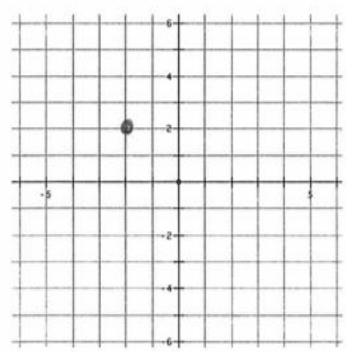
7.) $|-1-i|$ $\sqrt{1+1}$ $\boxed{\sqrt{2}}$



8.) $|-1+i|$ $\boxed{\sqrt{2}}$



9.) $|-2+2i|$ a b $\boxed{2\sqrt{2}}$



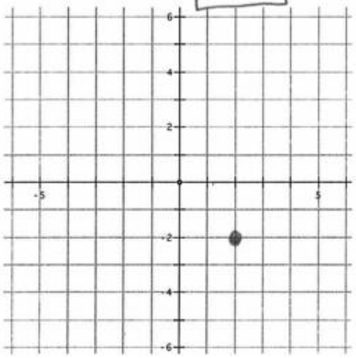
a + (b)i $\sqrt{a^2+b^2}$ $\sqrt{(-1)^2+(1)^2}$

-1 - i $\sqrt{(-1)^2+(-1)^2} = \sqrt{2}$ $\sqrt{1+1} = 2$

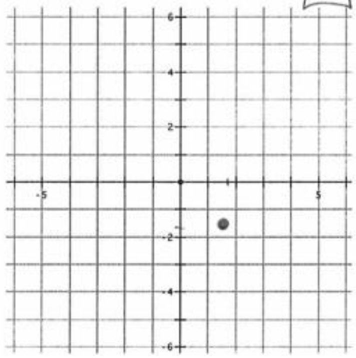
$\sqrt{(-2)^2+(2)^2} = \sqrt{8}$

$2 \cdot 4 \cdot 2 = 2\sqrt{2}$

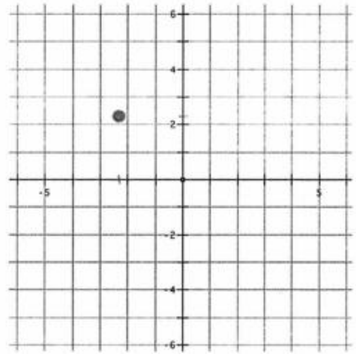
10.) $|2-2i|$ $\sqrt{a^2+b^2}$ $\sqrt{2^2+(-2)^2}$ $\sqrt{8}$ $\boxed{2\sqrt{2}}$



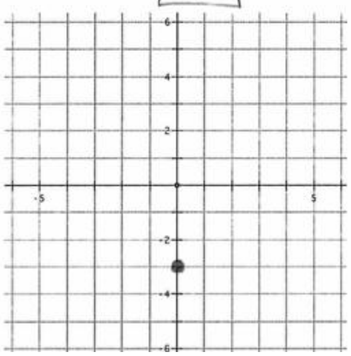
11.) $\sqrt{3}-i\sqrt{3}$ $\sqrt{(\sqrt{3})^2+(-\sqrt{3})^2}$ $\sqrt{3+3}$ $\sqrt{9}=\boxed{3}$



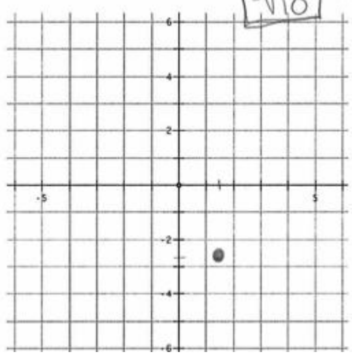
12.) $-\sqrt{5}+i\sqrt{5}$ $\sqrt{(-\sqrt{5})^2+\sqrt{5}^2}$ $\sqrt{5+5}$ $\sqrt{25}=\boxed{5}$



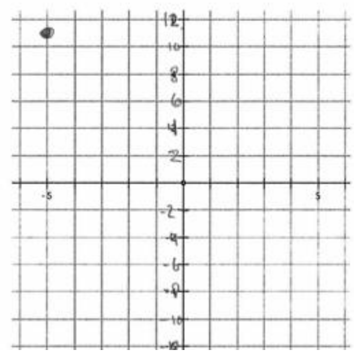
13.) $-3i$ $\sqrt{0^2+(-3)^2}=\sqrt{9}$ $0-3i$ $\boxed{3}$



14.) $\sqrt{2}-2i\sqrt{2}$ $\sqrt{(\sqrt{2})^2+(2\sqrt{2})^2}$ $\sqrt{2+8}=\sqrt{10}$ $\boxed{\sqrt{10}}$



15.) $-5+5i\sqrt{5}$ $\sqrt{(-5)^2+(5\sqrt{5})^2}$ $\sqrt{25+125}=\sqrt{150}$ $\boxed{5\sqrt{6}}$



Find each sum graphically. Check algebraically.

16.) $(4+i)+(-4+5i)$

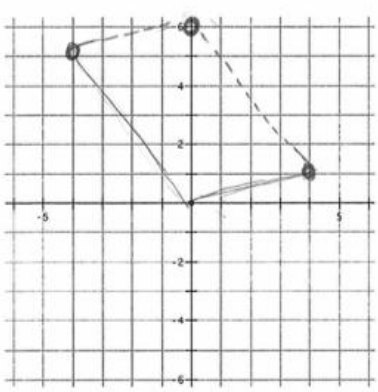
$=0+6i$

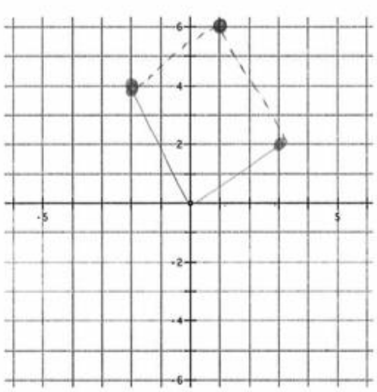
17.) $(3+2i)+(-2+4i)$

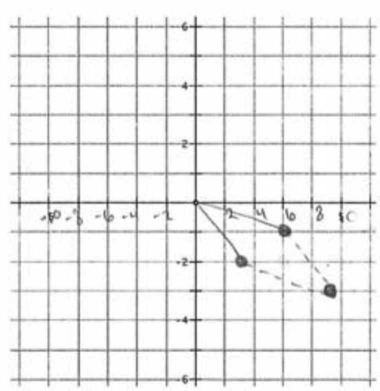
$=1+6i$

18.) $(6-i)+(3-2i)$

$=9-3i$







Reflection on Activity 3

Students found this lesson very easy and were expecting more difficult work. They adapted quickly to graphing on the complex plane. Several students commented on the similarities between finding the absolute value of a complex number and finding the magnitude of a vector. The only portion of this activity that gave students trouble was with calculations when a or b was a radical. Here I think it was how to graph a radical, and the look of a radical squared and underneath a radical when finding the absolute value.

Students needed prompting with the process for the last three questions. Once I suggested using a process similar to adding vectors they all were able to complete these exercises.
When I teach this unit again I will change the directions for problems 1 – 6 to be more specific. I was looking for an ordered pair to identify the location. Notice that the student work I have included shows these ordered pairs with a value for the vertical coordinate that includes an *i*.

Activity 4 Lesson Plan – Polar Coordinates

Instructor: Cynthia Schneider

Subject: Mathematics

Grade Level: 12th grade

Title: Polar Coordinates

Unit Title: Topics in Analytic Geometry

Content/Topic: This lesson provides students an additional mathematical perspective of graphing. Students will learn how to plot points in polar form. Students will convert points from rectangular to polar form, and polar to rectangular. Students will be introduced to the conversion of equations from rectangular to polar form and vice versa.

Content Objectives: Students will be able to plot and find multiple representations of points in the polar coordinate system. Students will be able to convert points from rectangular to polar form and vice versa.

Language Objectives: Students will understand and be able to use the following terminology: polar coordinate system, polar axis, pole,

Required materials:

- SMART Board lesson including definitions and examples from PRECALCULUS WITH LIMITS: A GRAPHING APPROACH, Third Edition
- Polar Coordinates worksheet #4
- Polar Graph paper for students to take notes on

Instruction and Practice: See included SMART Board Lesson

Time Allotment: Allow 30 – 35 minutes for the lesson, and 40 – 45 minutes for students to complete the Polar Coordinates worksheet #4.

Assessment: See included Polar Coordinates worksheet #4

Activity 4 SMART Board Slides – Polar Coordinates

Slide 1

Polar Coordinates

What you should learn:

How to plot points and find multiple representations of points in the polar coordinate system

How to convert points from rectangular to polar form and vice versa

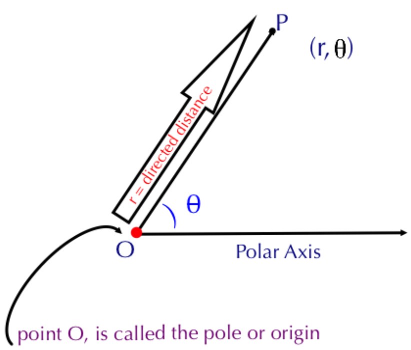
How to convert equations from rectangular to polar form and vice versa

Slide 2

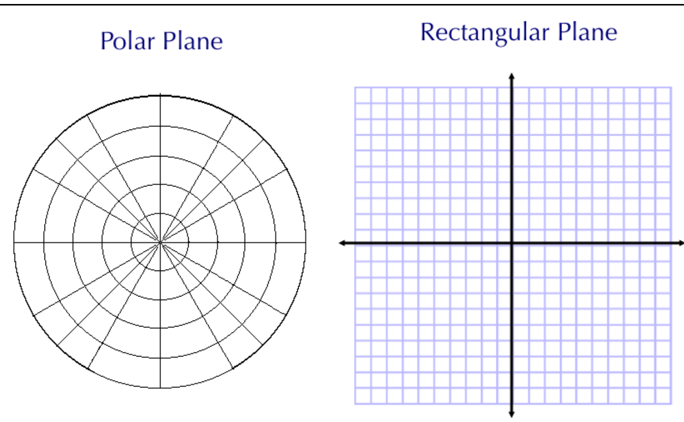
Each point on the plane can be assigned the ordered pair coordinate (r, θ) where

r is the directed distance from O to P

θ is the directed angle measured counterclockwise from the polar axis to segment PO.



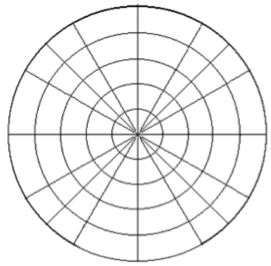
Slide 3



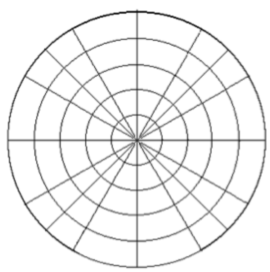
Each rectangular coordinate has a unique representation.
For polar coordinates this is not true.

Slide 4

Plot the point $\left(1, \frac{7\pi}{4}\right)$ in polar coordinates and find three additional polar representations of the point with $-2\pi \le \theta \le 2\pi$.



Plot the point $\left(2, \frac{-2\pi}{3}\right)$ in polar coordinates and find three additional polar representations of the point with $-2\pi \le \theta \le 2\pi$.



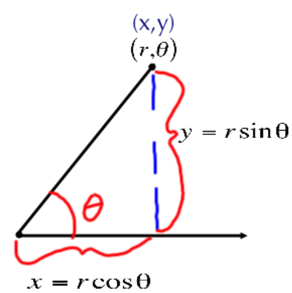
Slide 5

Coordinate Conversions:
The polar coordinates (r, θ) are related to the rectangular coordinates (x, y) as follows.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\tan\theta = \frac{y}{x} \qquad r^2 = x^2 + y^2$$

$$\theta = \tan^{-1}\left(\frac{y}{x}\right) \qquad r = \sqrt{x^2 + y^2}$$



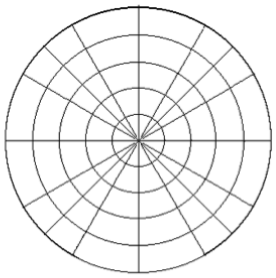
EX: Convert the point to rectangular coordinates.

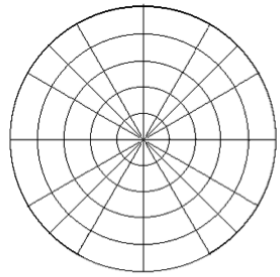
$$\left(2, \frac{3\pi}{4}\right) \qquad \left(-3, -\frac{2\pi}{3}\right)$$

Slide 6

EX: Plot the point given in polar coordinates and find the corresponding rectangular coordinates for the point.

$\left(-1, -\frac{3\pi}{4}\right)$ $\left(2, \frac{7\pi}{6}\right)$





Slide 7

EX: Given the rectangular coordinates (-3, -3) find two sets of polar coordinates for the point for $0 \leq \theta \leq 2\pi$.

EX: Given the rectangular coordinates (3, -1) find two sets of polar coordinates for the point for $0 \leq \theta \leq 2\pi$.

EX: Given the rectangular coordinates (5, 12) find two sets of polar coordinates for the point for $0 \leq \theta \leq 2\pi$.

Slide 8

We can convert rectangular equations to polar equations by using the coordinate definitions.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\tan\theta = \frac{y}{x} \qquad r^2 = x^2 + y^2$$

Slide 9

EX: Convert the rectangular equation to polar form.

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$\tan\theta = \frac{y}{x} \qquad r^2 = x^2 + y^2$$

$x^2 + y^2 = 64$

$x = 3$

$x^2 + y^2 - 6x = 0$

Slide 10

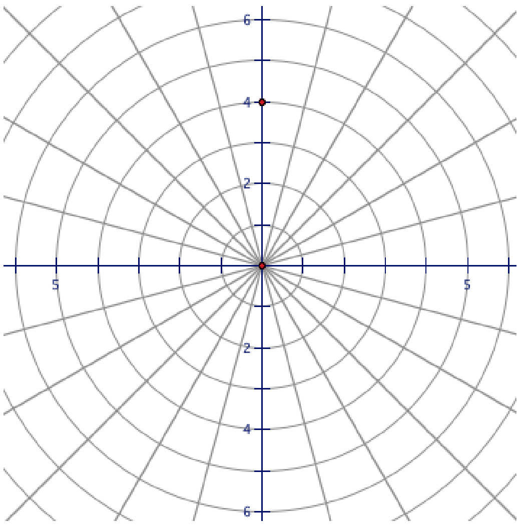
Assignment: Polar Coordinates worksheet

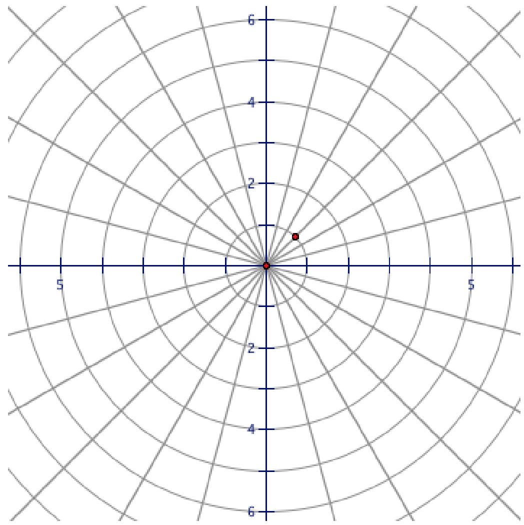
Activity 4 – Polar Coordinates

Find the corresponding rectangular coordinates for the given polar points.

1.) $\left(4,\ \frac{\pi}{2}\right)$

2.) $\left(-1,\ \frac{5\pi}{4}\right)$

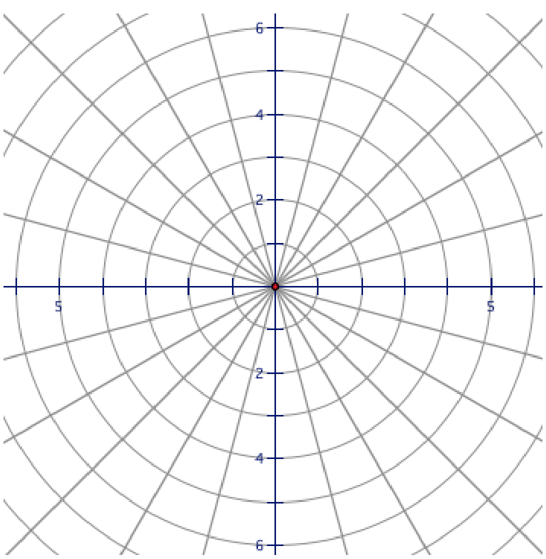


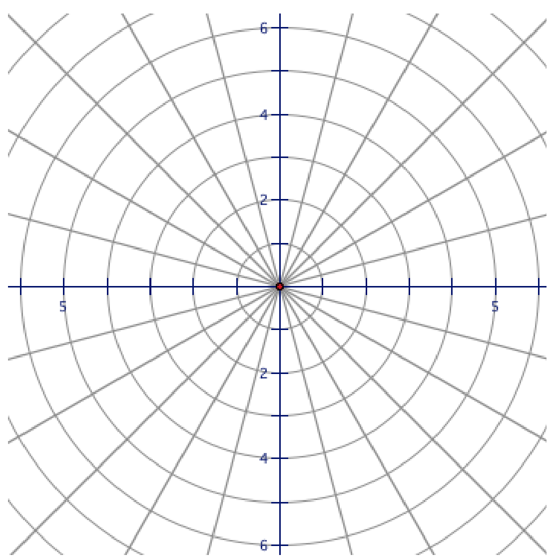


Plot the point given in polar coordinates and fin three additional polar representations of the point using $-2\pi < \theta < 2\pi$.

3.) $\left(4,\ \frac{2\pi}{3}\right)$

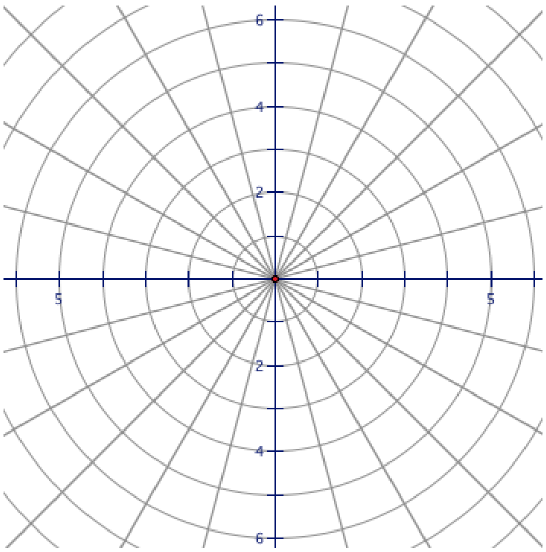
4.) $\left(5,\ -\frac{5\pi}{3}\right)$

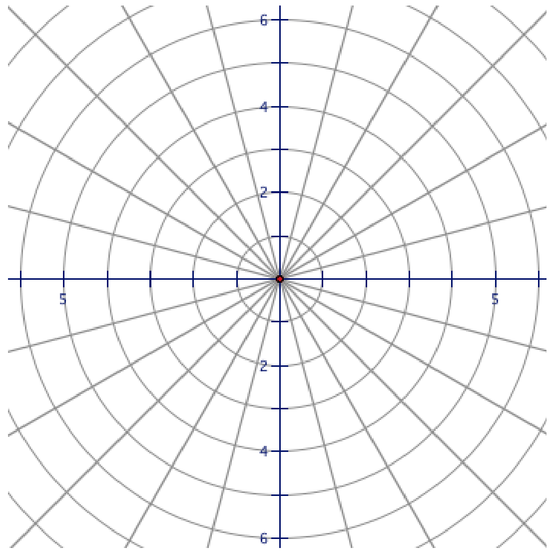




5.) $\left(\frac{3}{2},\ -\frac{3\pi}{2}\right)$

6.) $\left(-3,\ -\frac{7\pi}{6}\right)$

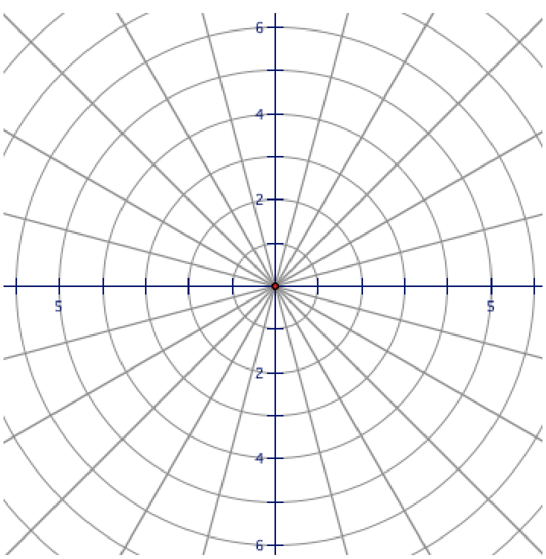


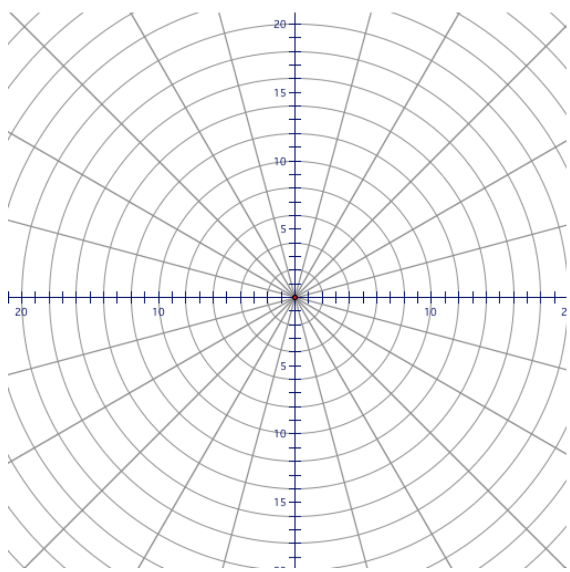


Plot the point given in polar coordinates and find the corresponding rectangular coordinates for the point.

7.) $\left(4,\ -\frac{\pi}{3}\right)$

8.) $\left(18,\ -\frac{3\pi}{2}\right)$

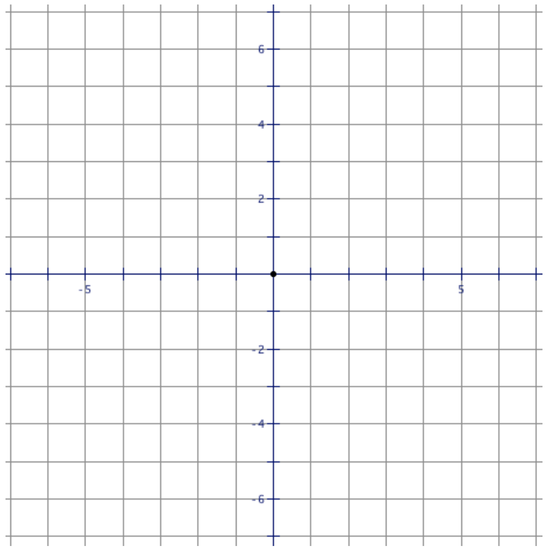


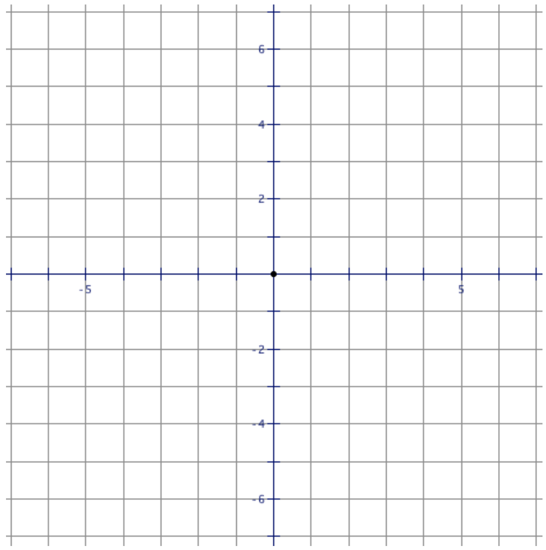


Plot the given rectangular coordinates and find two sets of polar coordinates for the point for $0 \le \theta < 2\pi$.

9.) $(-7,\ 0)$

10.) $(-3,\ 4)$





Convert the rectangular equation to polar form. Assume $a \ge 0$.

11.) $x^2 + y^2 = 49$

12.) $x^2 + y^2 = a^2$

13.) $x^2 + y^2 - 2ax = 0$

14.) $x^2 + y^2 - 2ay = 0$

15.) $x = 12$

16.) $x = a$

Convert the polar equation to rectangular form.

17.) $r = 4\sin\theta$

18.) $\theta = \dfrac{\pi}{6}$

19.) $r^2 = \cos\theta$

Activity 4 – Polar Coordinates (Teacher Version)

Find the corresponding rectangular coordinates for the given polar points.

1.) $\left(4, \frac{\pi}{2}\right)$

$x = 4\cos\frac{\pi}{2}$

$= 4(0)$

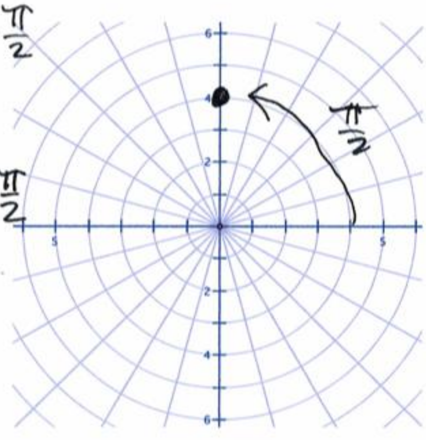
$= 0$

$y = 4\sin\frac{\pi}{2}$

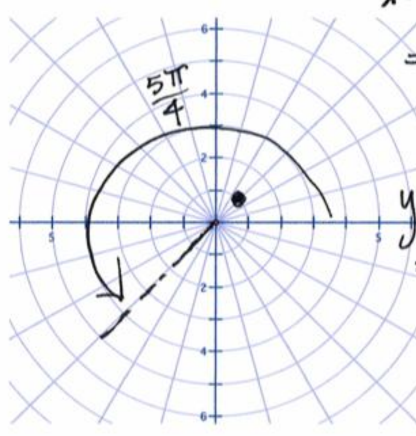
$= 4(1)$

$= 4$

$(0, 4)$



2.) $\left(-1, \frac{5\pi}{4}\right)$



$x = -1\cos\frac{5\pi}{4}$

$= -1\left(-\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{2}}{2}$

$y = -1\sin\frac{5\pi}{4}$

$= -1\left(-\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{2}}{2}$

$\left(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}\right)$

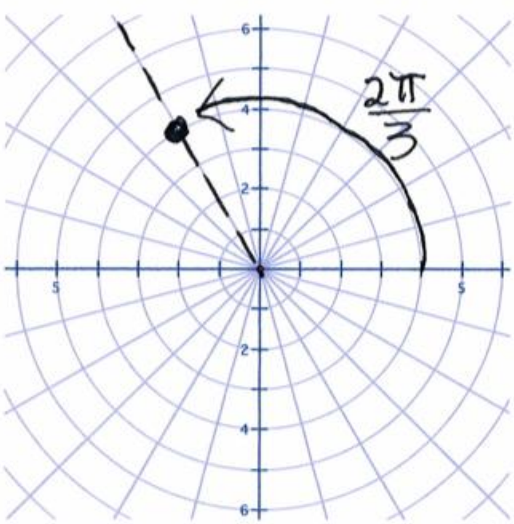
Plot the point given in polar coordinates and fin three additional polar representations of the point using $-2\pi < \theta < 2\pi$.

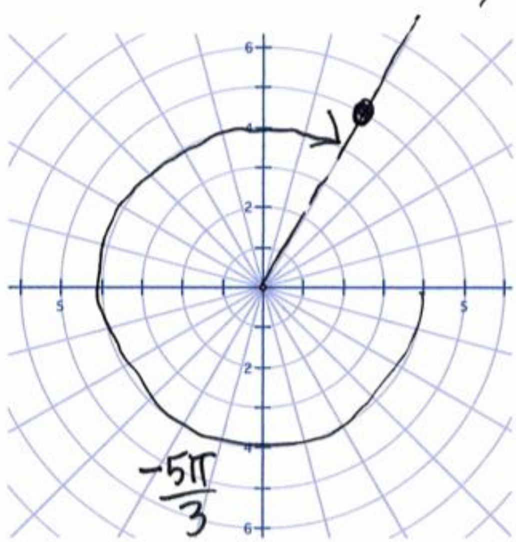
3.) $\left(4, \frac{2\pi}{3}\right)$ $\left(-4, -\frac{\pi}{3}\right)$

$\left(4, -\frac{4\pi}{3}\right)$ $\left(-4, \frac{5\pi}{3}\right)$

4.) $\left(5, -\frac{5\pi}{3}\right)$ $\left(5, \frac{\pi}{3}\right)$

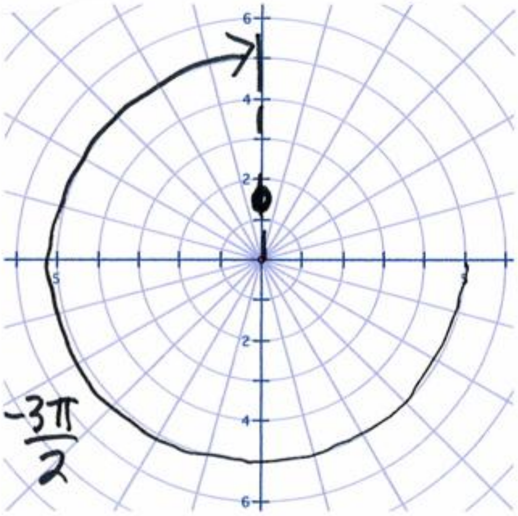
$\left(-5, -\frac{2\pi}{3}\right)$ $\left(-5, \frac{4\pi}{3}\right)$





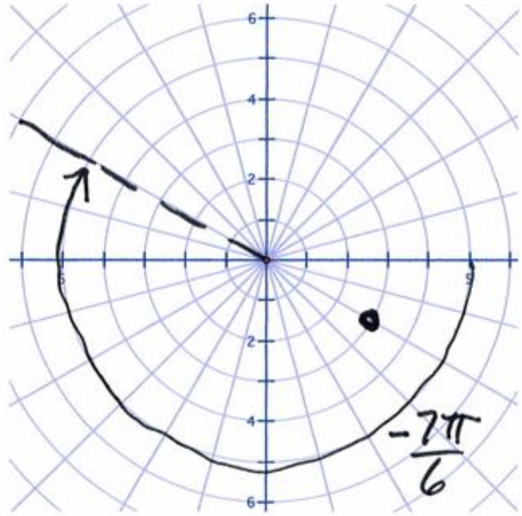
5.) $\left(\frac{3}{2}, -\frac{3\pi}{2}\right)$ $\left(\frac{3}{2}, \frac{\pi}{2}\right)$

$\left(-\frac{3}{2}, -\frac{\pi}{2}\right)$ $\left(-\frac{3}{2}, \frac{3\pi}{2}\right)$



6.) $\left(-3, -\frac{7\pi}{6}\right)$ $\left(3, -\frac{\pi}{6}\right)$

$\left(3, \frac{11\pi}{6}\right)$ $\left(-3, \frac{5\pi}{6}\right)$



Plot the point given in polar coordinates and find the corresponding rectangular coordinates for the point.

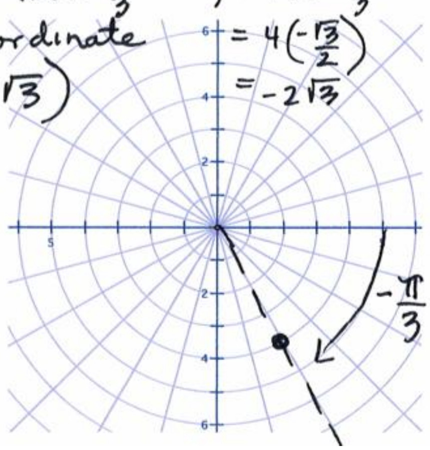
7.) $\left(4, -\frac{\pi}{3}\right)$

$x = 4\cos\frac{5\pi}{3}$
$= 4\left(\frac{1}{2}\right)$
$= 2$

$\frac{5\pi}{3}$ coterminal with $-\frac{\pi}{3}$

$y = 4\sin\frac{5\pi}{3}$
$= 4\left(-\frac{\sqrt{3}}{2}\right)$
$= -2\sqrt{3}$

Rect. Coordinate
$(2, -2\sqrt{3})$



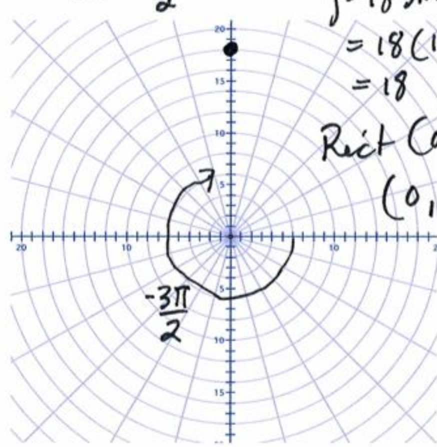
8.) $\left(18, -\frac{3\pi}{2}\right)$

$x = 18\cos\frac{\pi}{2}$
$= 18(0)$
$= 0$

$-\frac{3\pi}{2}$ coterminal with $\frac{\pi}{2}$

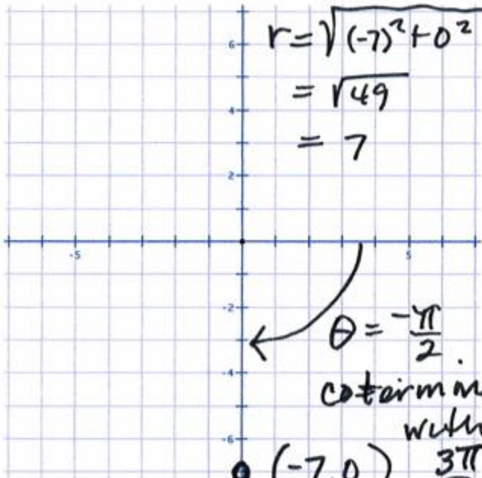
$y = 18\sin\frac{\pi}{2}$
$= 18(1)$
$= 18$

Rect Coord:
$(0, 18)$



Plot the given rectangular coordinates and find two sets of polar coordinates for the point for $0 \le \theta < 2\pi$.

9.) $(-7, 0)$ $\left(7, \frac{3\pi}{2}\right)$



$r = \sqrt{(-7)^2 + 0^2}$
$= \sqrt{49}$
$= 7$

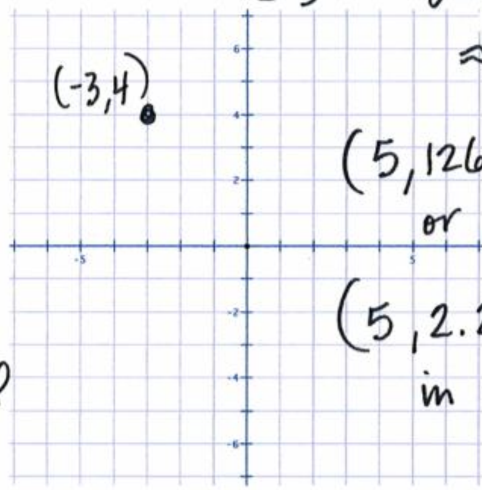
$\theta = -\frac{\pi}{2}$.
coterminal with $\frac{3\pi}{2}$

$(-7, 0)$

10.) $(-3, 4)$

$r = \sqrt{(-3)^2 + (4)^2}$
$= \sqrt{25}$
$= 5$

$\theta = \tan^{-1}\left(\frac{4}{-3}\right) + \pi$
$\approx 126.9^\circ$



$(-3, 4)$

$(5, 126.9^\circ)$
or
$(5, 2.2)$
in radians

$x^2 + y^2 = r^2$
$x = r\cos\theta$
$y = r\sin\theta$

Convert the rectangular equation to polar form. Assume $a \ge 0$.

11.) $x^2 + y^2 = 49$

$r^2 = 49$

$\boxed{r = 7}$

circle of radius 7

12.) $x^2 + y^2 = a^2$

$r^2 = a^2$

$\boxed{r = a}$

circle of radius a

13.) $x^2 + y^2 - 2ax = 0$

$r^2 - 2ar\cos\theta = 0$

$r(r - 2a\cos\theta) = 0$

$r = 0$ $\quad r - 2a\cos\theta = 0$ or $\boxed{r = 2a\cos\theta}$

14.) $x^2 + y^2 - 2ay = 0$

$r^2 - 2ar\sin\theta = 0$

$r(r - 2a\sin\theta) = 0$

$r = 0$ $\quad r - 2a\sin\theta = 0$

$\boxed{r = 2a\sin\theta}$

15.) $x = 12$

$r\cos\theta = 12$

$\boxed{r = \frac{12}{\cos\theta}}$

16.) $x = a$

$r\cos\theta = a$

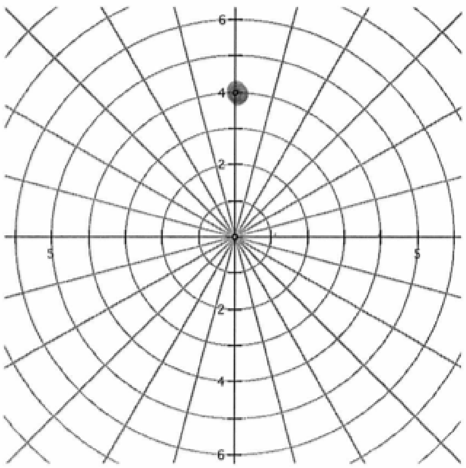
$\boxed{r = \frac{a}{\cos\theta}}$

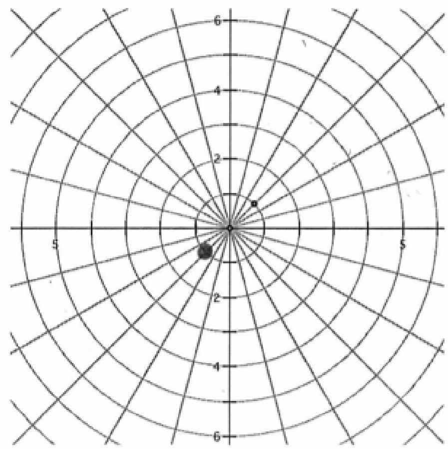
Activity 4 – Polar Coordinates (Student Work)

Find the corresponding rectangular coordinates for the given polar points.

1.) $\left(4, \frac{\pi}{2}\right)$

2.) $\left(-1, \frac{5\pi}{4}\right)$



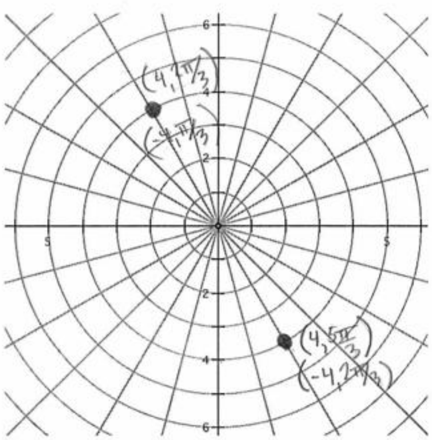


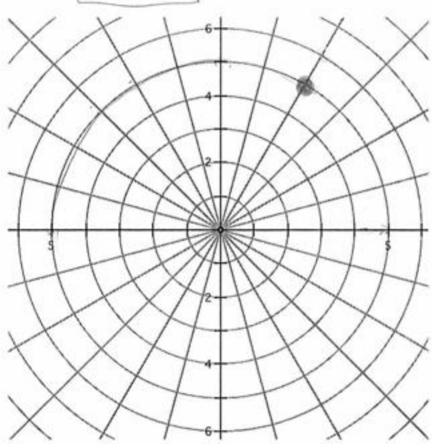
Plot the point given in polar coordinates and fin three additional polar representations of the point using $-2\pi < \theta < 2\pi$.

3.) $\left(4, \frac{2\pi}{3}\right)$ $\left(4, \frac{5\pi}{3}\right)$ $+\pi$ $\left(-4, -\frac{\pi}{3}\right)$ $\left(-4, \frac{2\pi}{3}\right)$

4.) $\left(5, -\frac{5\pi}{3}\right)$ $\left(-5, -\frac{2\pi}{3}\right)$ $\left(5, \frac{\pi}{3}\right)$ $\left(-5, \frac{4\pi}{3}\right)$

$-\frac{5\pi}{3} + \frac{3\pi}{3} = -\frac{2\pi}{3}$ $\frac{5\pi}{3} = \frac{\pi}{3}$



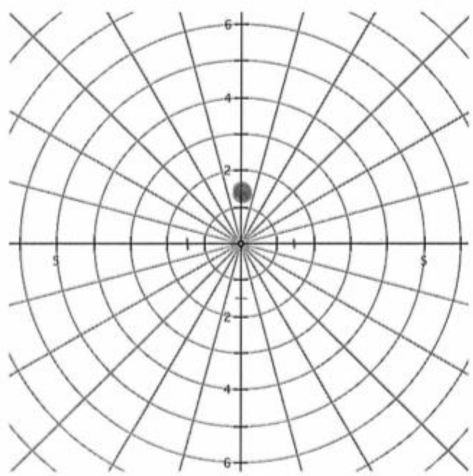


$\frac{\pi}{2} + \frac{\pi}{6}$

$\frac{3\pi}{6} + \frac{\pi}{6} = \frac{4\pi}{6} = \frac{2\pi}{3}$

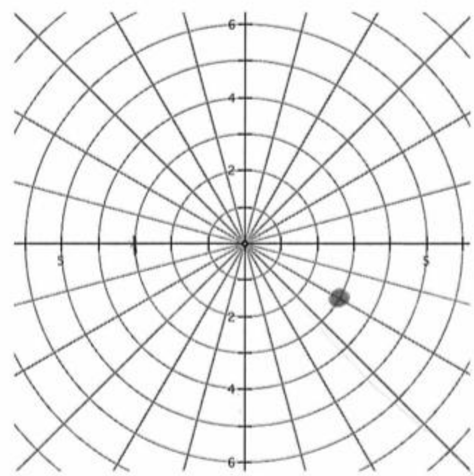
5.) $\left(\frac{3}{2}, -\frac{3\pi}{2}\right)$ $\left(\frac{3}{2}, \frac{\pi}{2}\right)$

$\left(-\frac{3}{2}, \frac{3\pi}{2}\right)$ $\left(-\frac{3}{2}, -\frac{\pi}{2}\right)$



6.) $\left(-3, -\frac{7\pi}{6}\right)$ $\left(3, \frac{11\pi}{6}\right)$

$\left(-3, -\frac{5\pi}{6}\right)$ $\left(3, -\frac{\pi}{6}\right)$



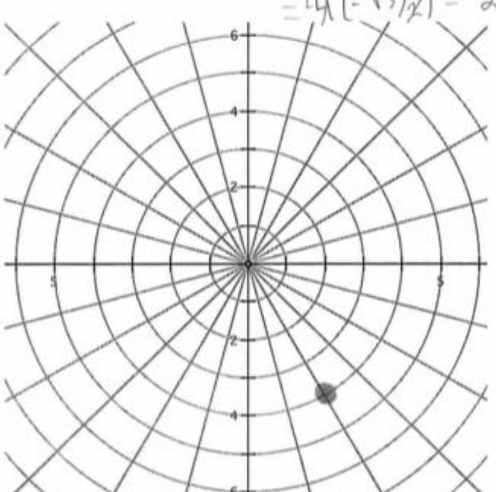
Plot the point given in polar coordinates and find the corresponding rectangular coordinates for the point.

7.) $\left(\overset{r}{4}, \overset{\theta}{-\frac{\pi}{3}}\right)$

$-\frac{\pi}{3} + 2\pi = \frac{5\pi}{3}$

$x = r\cos\theta$
$= 4\cos\frac{5\pi}{3}$
$= 4\left(\frac{1}{2}\right) = 2$

$y = r\sin\theta$
$= 4\sin\frac{5\pi}{3}$
$= 4\left(-\frac{\sqrt{3}}{2}\right) = -2\sqrt{3}$

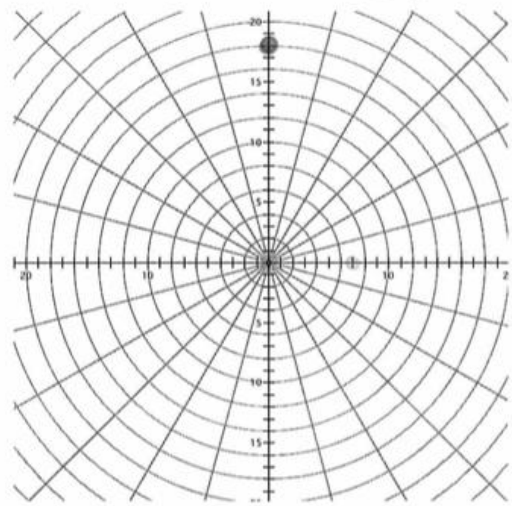


$\boxed{(2, -2\sqrt{3})}$

8.) $\left(\overset{r}{18}, \overset{\theta}{-\frac{3\pi}{2}}\right)$

$x = r\cos\theta$
$= 18\cos\frac{\pi}{2}$
$= 18(0) = 0$

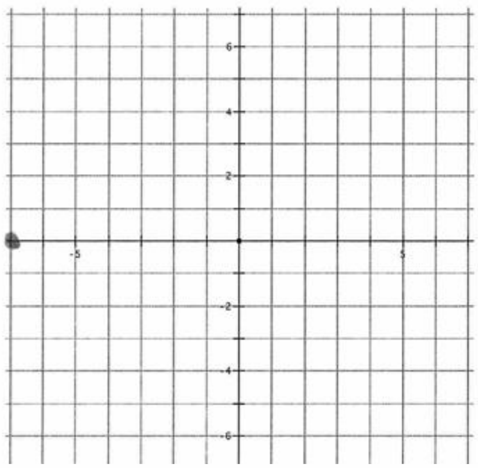
$y = r\sin\theta$
$= 18\sin\frac{\pi}{2}$
$= 18(1) = 18$



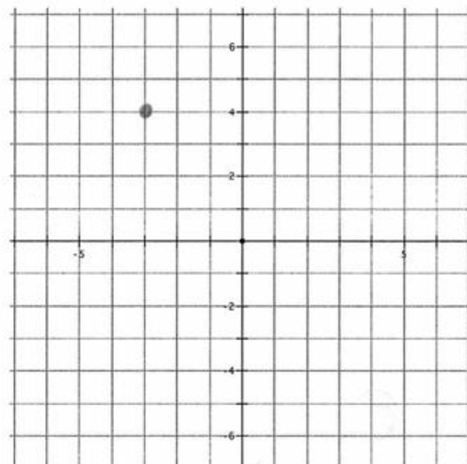
$\boxed{(0, 18)}$

Plot the given rectangular coordinates and find two sets of polar coordinates for the point for $0 \le \theta < 2\pi$.

9.) $(-7, 0)$ $r = \sqrt{(-7)^2 + 0^2} = 7$ $\theta = \tan^{-1}\left(\frac{0}{-7}\right) = 0 + \pi$ $\boxed{(7, \pi)}$ $\boxed{(-7, 0)}$



10.) $(-3, 4)$ $r = \sqrt{(-3)^2 + (4)^2} = \sqrt{25} = 5$ $\theta = \tan^{-1}\left(\frac{4}{-3}\right) \approx -53.1$ $(5, -53.1)$ $+180$ $\boxed{(5, 126.9)}$ $\boxed{(-5, -53.1)}$



Convert the rectangular equation to polar form. Assume $a \ge 0$.

11.) $x^2 + y^2 = 49$
$r^2 = 49$
$r = \sqrt{49} = 7$
$\boxed{r = 7}$

12.) $x^2 + y^2 = a^2$
$r^2 = x^2 + y^2$
$x + y = a$
$\theta = \tan^{-1}\left(\frac{y}{x}\right)$
$r = a$ $r^2 = a^2$
$\boxed{r = a}$

13.) $x^2 + y^2 - 2ax = 0$
$r^2 - 2a(r\cos\theta) = 0$
$r^2 - 2ar\cos\theta = 0$
$r^2 = 2ar\cos\theta$
$\boxed{r = 2a\cos\theta}$

14.) $x^2 + y^2 - 2ay = 0$
$r^2 - 2ay = 0$
$r^2 = 2a(r\sin\theta)$
$r^2 = 2ar\sin\theta$
$\boxed{r = 2a\sin\theta}$

15.) $x = 12$
$r\cos\theta = 12$
$\boxed{r = \frac{12}{\cos\theta}}$

16.) $x = a$
$r\cos\theta = a$
$\boxed{r = \frac{a}{\cos\theta}}$

Convert the polar equation to rectangular form.

17.) $(r)\, r^2 = 4\sin\theta\,(r)$
$r^2 = 4\sin\theta\, r$
$r^2 = 4y$
$\boxed{x^2 + y^2 = 4y}$

18.) $\theta = \frac{\pi}{6}$
$\boxed{y = \frac{\sqrt{3}}{3}x}$

$\frac{1}{2} \cdot \frac{2}{\sqrt{3}} = \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$

$y = mx + b$ $\frac{\sqrt{3}}{2}$

19.) $r^2 = \cos\theta$
$x^2 + y^2 = \cos\theta$

Reflection on Activity 4

Students had only minor problems plotting any given polar coordinate. However, most struggled with finding additional polar coordinates describing the same point. Note problem #4 in the student version of the activity. This student, like most, had trouble visualizing a rotation from the positive pole in the counterclockwise direction to a co-terminal angle. Every student had trouble finding the angle π units away with a negative r-value. I found the best way to help students think through this process was with a physical example. I would stand "on the pole" with my arm extended and say "I am pointing at the pole, or the positive x-axis". Then I would rotate my body $\theta = \frac{-\pi}{3}$ units clockwise. I would describe that I was now facing this direction θ and then take 4 steps out to the location. I would then repeat this process rotating in the counterclockwise direction $\theta = \frac{2\pi}{3}$. I would describe how I was facing opposite the location desired and so would take 4 steps backwards, represented by a negative value. For some reason this practice of physically moving to demonstrate the process seemed to help more students grasp the process of finding additional representations. Maybe it was just funny to watch their teacher spin around in circles?

Converting polar coordinates to rectangular was an easy process for students. Converting rectangular coordinates to polar coordinates was more difficult. Students had trouble with inverse tangent function value when the rectangular coordinates were in the 2^{nd} and 3^{rd} quadrant. I asked students to make a quick sketch of the point in rectangular coordinates. Then we discussed the inverse tangent function and its range. We would discuss the value of the inverse tangent function and how it related to the actual point. It took working through both problems before students began to see that they needed to add or subtract π to find the correct angle. I will include several more of these types of problems in the future.

Converting equations was extremely challenging. This was not major objective for this lesson but rather an introduction to the concept. With practice about half of my students could convert a basic equation from rectangular to polar. The reverse was much more difficult.

Activity 5 – Trigonometric Form of a Complex Number

Instructor: Cynthia Schneider

Subject: Mathematics

Grade Level: 12^{th} grade

Title: Trigonometric Form of a Complex Number

Unit Title: Additional Topics in Trigonometry

Content/Topic: This lesson is designed to instruct students on how to re-write complex numbers in polar form. Students will then learn to multiply and divide complex numbers in polar form. Students will use De Moivre's Theorem to find powers of complex numbers. Finally, students find nth roots of real and complex numbers.

Content Objectives: Students will be able to write complex numbers in polar or trigonometric form. Students will be able to multiply and divide complex numbers in polar or trigonometric form. Students will be able to use De Moivre's Theorem to find powers complex numbers and nth roots of real and complex numbers.

Language Objectives: Students will understand and be able to use the following terminology: trigonometric form of complex number, modulus, argument, De Moivre's Theorem, nth roots, and roots of unity.

Required materials:

- SMART Board lesson including definitions and examples from PRECALCULUS WITH LIMITS: A GRAPHING APPROACH, Third Edition
- Products and Quotients of Complex Numbers in Trig Form worksheet #5a
- De Moivre's Theorem and Nth Roots worksheet #5b

Instruction and Practice: See included SMART Board Lesson

Time Allotment: Allow two class periods to complete this lesson. On the first day allow 25 – 35 minutes for the lesson and 40 – 45 minutes to complete the worksheet #5a. On the second day allow 30 – 35 minutes for the lesson, and 40 – 45 minutes to complete worksheet #5b.

Assessment: See included worksheets #5a, and #5b.

Activity 5a SMART Board Slides – Products and Quotients of Complex Numbers in Trigonometric Form

Slide 1

Trigonometric Form of a Complex Number
PART ONE

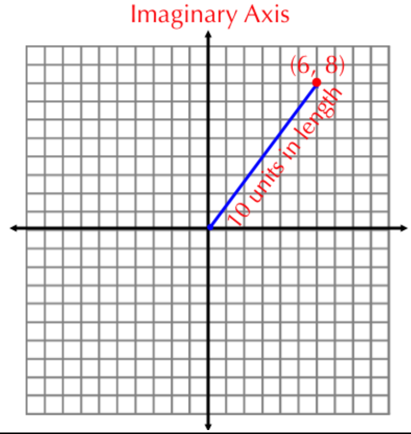
What you should learn:

How to find the absolute value of a complex number
How to write trigonometric forms of complex number
How to multiply and divide complex numbers in trig form

Slide 2

Absolute Value = magnitude or length

For 6 + 8*i*, $|(6+8i)| = \sqrt{6^2+8^2} = \sqrt{36+64} = \sqrt{100} = 10$



Slide 3

The trigonometric form of a complex number is also the polar form of a complex number.
Then for any complex number z = a + b*i* with complex coordinates (a, b) we use the directed line segment from the origin to (a, b) and the angle θ from the polar axis to the directed line segment to describe a + b*i*.

$$a = r\cos\theta \qquad b = r\sin\theta$$
$$r = \sqrt{a^2+b^2} \qquad \tan\theta = \frac{b}{a}$$

Then $z = a + bi = (r\cos\theta) + (r\sin\theta)i$

or

$$z = r(\cos\theta + i\sin\theta)$$

where r is the **modulus** of z, and θ is called an **argument** of z.

Slide 4

Remember that polar coordinates are not unique.
Normally, θ is restricted to the interval $0 \le \theta \le 2\pi$,
but is may be convenient to use $0 < \theta$.

Slide 5

Write the complex number in trigonometric form.

$$z = 3 - 4i$$

$$z = 1 + 3i$$

$$z = 2\sqrt{2} - i$$

Slide 6

Write the complex number in standard form.

$$z = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) \qquad z = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$$

Slide 7

Multiplication of Complex Numbers in Trig Form

Given

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$z_1 z_2 =$$

Slide 8

Division of Complex Numbers in Trig Form

Given

$$z_1 = r_1(\cos\theta_1 + i\sin\theta_1) \qquad z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$$

$$\frac{z_1}{z_2} =$$

Slide 9

Find the product and the quotient, and leave the result in trigonometric form.

$z_1 = 40\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$ $z_2 = 5\left(\cos\frac{3\pi}{5} + i\sin\frac{3\pi}{5}\right)$

Slide 10

Find the product and the quotient, and leave the result in trigonometric form.

$z_1 = 24i$ $z_2 = 4\sqrt{3} - 4i$

Slide 11

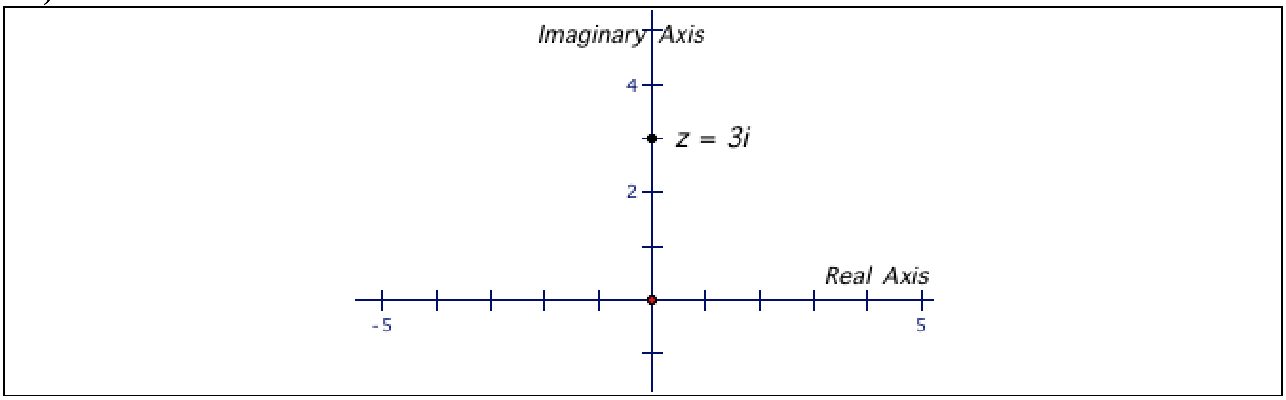
☆

Product and Quotient of Complex Numbers worksheet

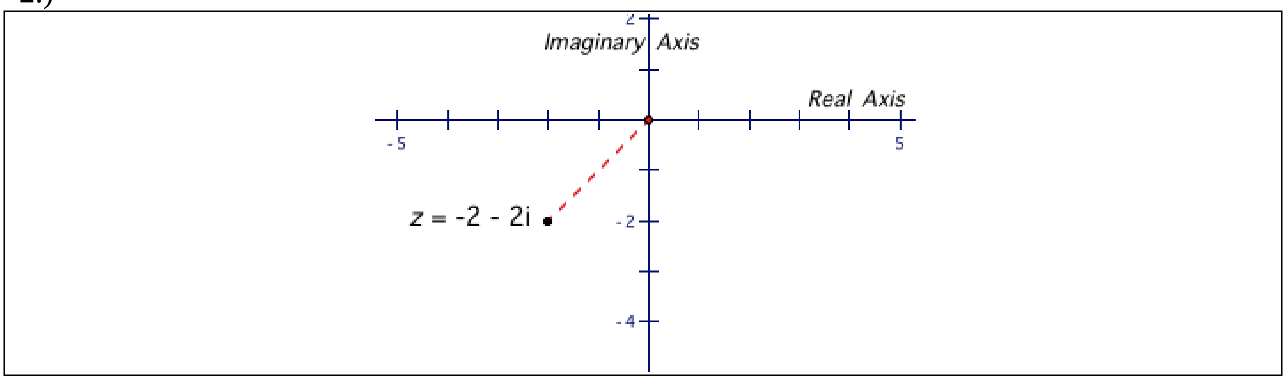
Activity 5a - Products and Quotients of Complex Numbers in Trig Form

Write the complex number in trigonometric form.

1.)



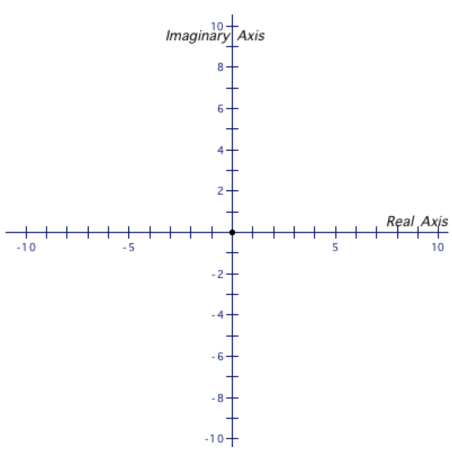
2.)

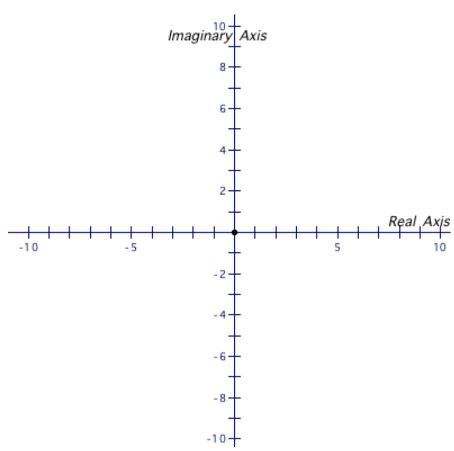


Represent the complex number graphically, and find the trigonometric form of the number.

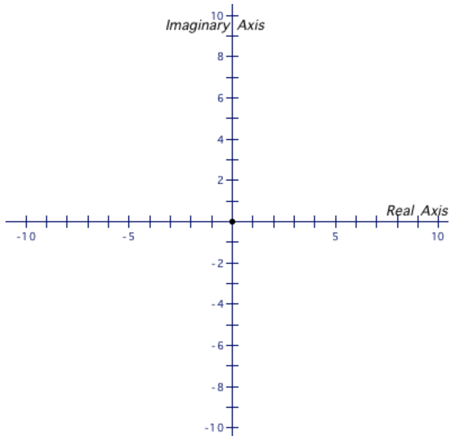
3.) $5-5i$

4.) $\sqrt{3}+i$

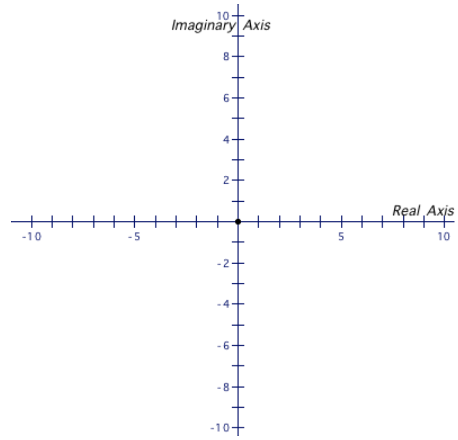




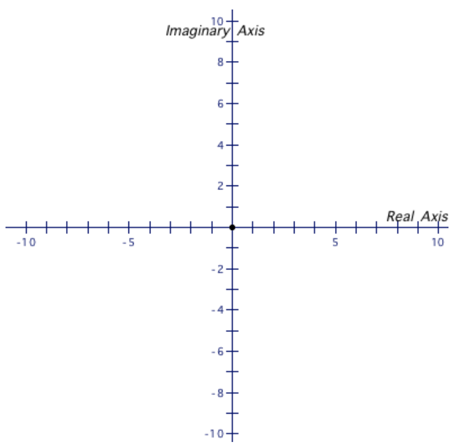
5.) $-2\left(1+i\sqrt{3}\right)$



6.) $8i$



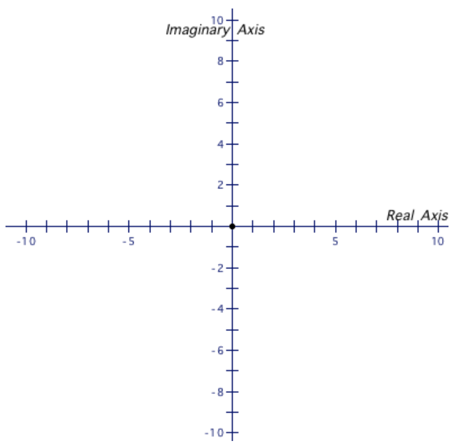
7.) $-7+4i$

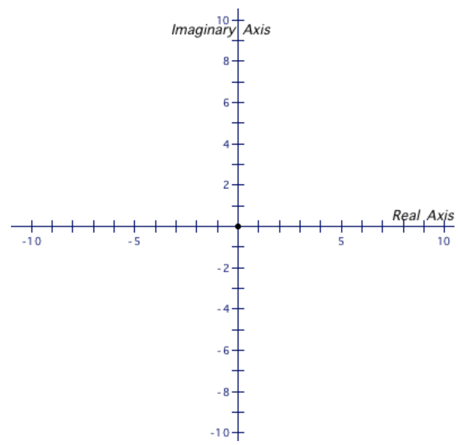


Represent the complex number graphically, and find the standard form of the number.

8.) $2(\cos 120° + i \sin 120°)$

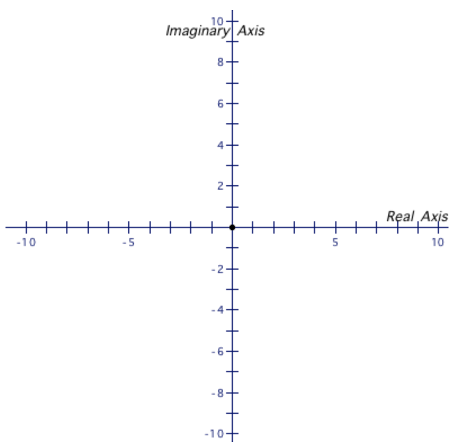
9.) $\frac{3}{2}(\cos 330° + i \sin 330°)$

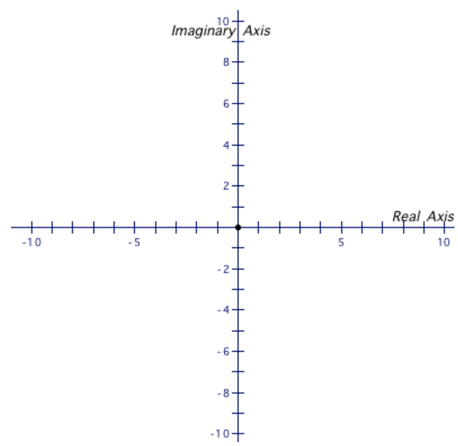




10.) $3.75\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$

11.) $4\left(\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}\right)$





Perform the operation and leave the result in trigonometric form.

12.) $\left[3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]\left[4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]$

13.) $\left[\frac{3}{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]\left[6\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]$

14.) $\left[\frac{5}{3}(\cos 140^\circ+i\sin 140^\circ)\right]\left[\frac{2}{3}(\cos 60^\circ+i\sin 60^\circ)\right]$

15.) $(\cos 5^\circ+i\sin 5^\circ)(\cos 20^\circ+i\sin 20^\circ)$

16.) $\dfrac{\cos 50^\circ+i\sin 50^\circ}{\cos 20^\circ+i\sin 20^\circ}$

17.) $\dfrac{2(\cos 120^\circ+i\sin 120^\circ)}{4(\cos 40^\circ+i\sin 40^\circ)}$

18.) $$\frac{\cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)}{\cos\pi+i\sin\pi}$$

19.) $$\frac{18(\cos 54°+i\sin 54°)}{3(\cos 102°+i\sin 102°)}$$

20.) $$\frac{9(\cos 20°+i\sin 20°)}{5(\cos 75°+i\sin 75°)}$$

You have in your notes the proof for multiplying complex numbers in trigonometric form. Use a similar process to prove the following.

21.)

Given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $z_2 \neq 0$,

Prove that $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2)\right]$.

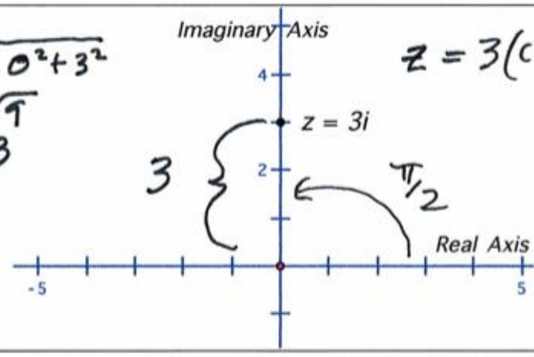
Activity 5a - Products and Quotients of Complex Numbers in Trig Form

(Teacher Version)

Write the complex number in trigonometric form.

1.)

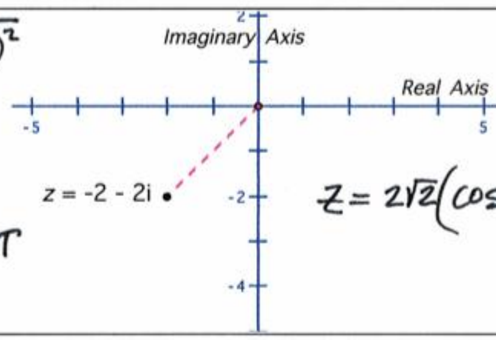
$|z| = \sqrt{0^2+3^2}$
$= \sqrt{9}$
$= 3$



$z = 3\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

2.)

$|z| = \sqrt{(-2)^2+(-2)^2}$
$= \sqrt{8}$
$= 2\sqrt{2}$
$\theta = \tan^{-1}\left(\frac{-2}{-2}\right) + \pi$
$\theta = \frac{5\pi}{4}$



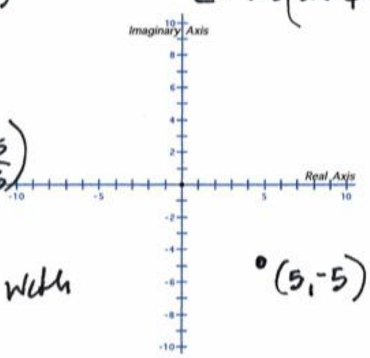
$z = 2\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$

Represent the complex number graphically, and find the trigonometric form of the number.

3.) $z = 5 - 5i$

$r = \sqrt{5^2+(-5)^2}$
$= \sqrt{50}$
$= 5\sqrt{2}$
$\theta = \tan^{-1}\left(\frac{-5}{5}\right)$
$= \frac{-\pi}{4}$
coterminal with
$\frac{7\pi}{4}$

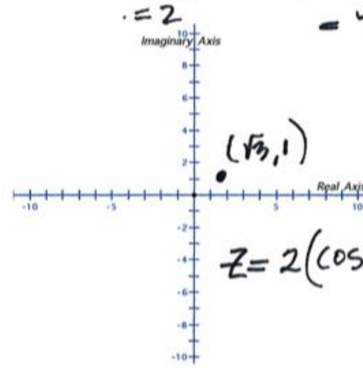
$z = 5\sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right)$



4.) $z = \sqrt{3} + i$

$r = \sqrt{(\sqrt{3})^2+1^2}$
$= \sqrt{4}$
$= 2$

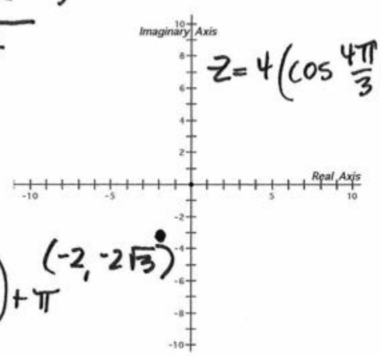
$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$
$= \pi/6$



$z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

5.) $z = -2\left(1+i\sqrt{3}\right) = -2 - 2i\sqrt{3}$

$r = \sqrt{(-2)^2 + (-2\sqrt{3})^2}$
$= \sqrt{4+12}$
$= \sqrt{16}$
$= 4$



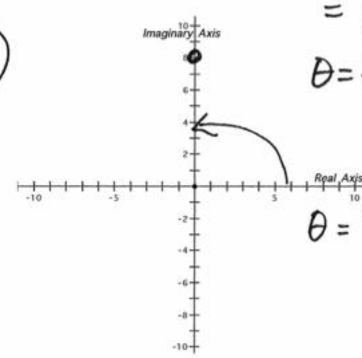
$z = 4\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$\theta = \tan^{-1}\left(\frac{-2\sqrt{3}}{-2}\right) + \pi$
$= \frac{4\pi}{3}$

6.) $8i$

$r = \sqrt{8^2 + 0^2}$
$= 8$

$\theta = \tan^{-1}\left(\frac{8}{0}\right)$
undefine

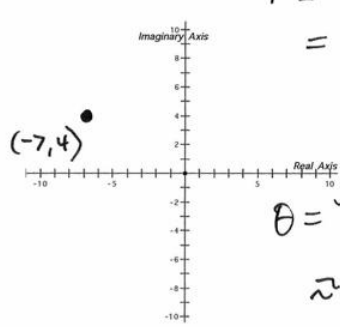


$\theta = \frac{\pi}{2}$

$z = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

7.) $-7+4i$

$r = \sqrt{(-7)^2 + 4^2}$
$= \sqrt{65}$



$\theta = \pi + \tan^{-1}\left(\frac{4}{-7}\right)$
$\approx 150.26^\circ$
≈ 2.62 radians

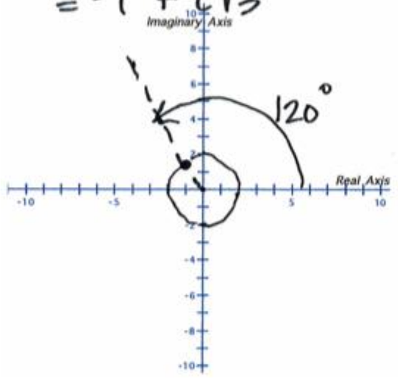
$z = \sqrt{65}\left(\cos 2.62 + i\sin 2.62\right)$

Represent the complex number graphically, and find the standard form of the number.

8.) $2(\cos 120° + i \sin 120°)$

$= 2\left(-\frac{1}{2}\right) + i\,2\left(\frac{\sqrt{3}}{2}\right)$

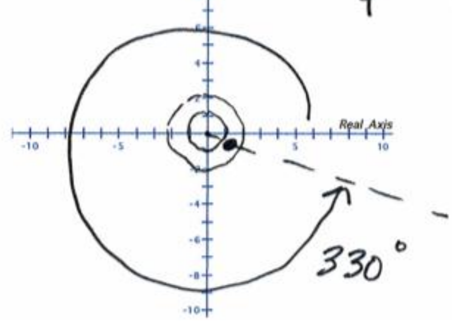
$= -1 + i\sqrt{3}$



9.) $\frac{3}{2}(\cos 330° + i \sin 330°)$

$= \frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) + i\,\frac{3}{2}\left(-\frac{1}{2}\right)$

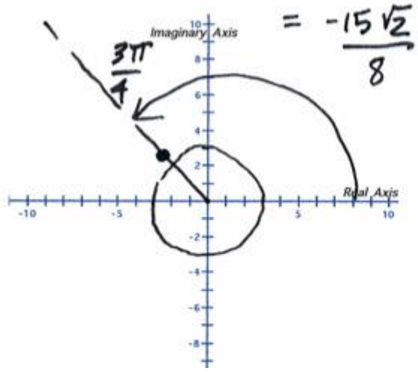
$= \frac{3\sqrt{3}}{4} - \frac{3i}{4}$



10.) $3.75\left(\cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4}\right)$

$= 3.75\left(-\frac{\sqrt{2}}{2}\right) + i\,3.75\left(\frac{\sqrt{2}}{2}\right)$

$= -\frac{15\sqrt{2}}{8} + \frac{15i\sqrt{2}}{8}$

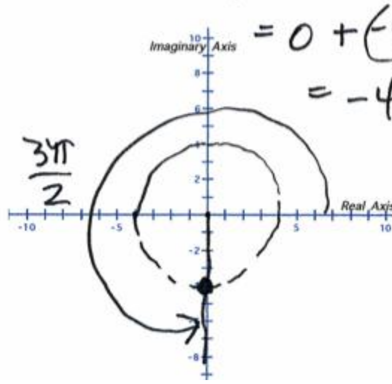


11.) $4\left(\cos\frac{3\pi}{2} + i \sin\frac{3\pi}{2}\right)$

$= 4(0) + 4i(-1)$

$= 0 + (-4i)$

$= -4i$



Perform the operation and leave the result in trigonometric form.

12.) $\left[3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]\left[4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]$

$= 3(4)\left(\cos \frac{\pi}{3}+\frac{\pi}{6} + i\sin\frac{\pi}{3}+\frac{\pi}{6}\right)$

$= 12\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)$

13.) $\left[\frac{3}{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]\left[6\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]$

$= \frac{3}{2}(6)\left[\cos\frac{\pi}{6}+\frac{\pi}{4} + i\sin\frac{\pi}{6}+\frac{\pi}{4}\right]$

$= 9\left(\cos\frac{5\pi}{12}+i\sin\frac{5\pi}{12}\right)$

14.) $\left[\frac{5}{3}(\cos 140° + i\sin 140°)\right]\left[\frac{2}{3}(\cos 60° + i\sin 60°)\right]$

$= \frac{5}{3}\left(\frac{2}{3}\right)\left[\cos 140+60 + i\sin 140+60\right]$

$= \frac{10}{9}(\cos 200° + i\sin 200)$

15.) $(\cos 5° + i\sin 5°)(\cos 20° + i\sin 20°)$

$\cos 5+20 + i\sin 5+20$

$= \cos 25° + i\sin 25°$

16.) $\dfrac{\cos 50° + i\sin 50°}{\cos 20° + i\sin 20°}$

$= \cos 50-20 + i\sin 50-20$

$= \cos 30° + i\sin 30°$

17.) $\dfrac{2(\cos 120° + i\sin 120°)}{4(\cos 40° + i\sin 40°)}$

$= \frac{2}{4}\left[\cos 120-40 + i\sin 120-40\right]$

$= \frac{1}{2}(\cos 80° + i\sin 80°)$

18.) $\dfrac{\cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)}{\cos\pi+i\sin\pi}$

$= \cos \frac{7\pi}{4}-\pi + i \sin\frac{7\pi}{4}-\pi$

$= \cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4}$

19.) $\dfrac{18(\cos 54° + i\sin 54°)}{3(\cos 102° + i\sin 102°)}$

$= \frac{18}{3}(\cos 54-102 + i \sin 54-102)$

$= 6(\cos -48 + i \sin -48)$

$-48°$ coterminal with $312°$

$= 6(\cos 312° + i \sin 312°)$

20.) $\dfrac{9(\cos 20° + i\sin 20°)}{5(\cos 75° + i\sin 75°)}$

$= \frac{9}{5}(\cos 20-75 + i \sin 20-75)$

$= \frac{9}{5}(\cos -55 + i \sin -55)$

-55 coterminal with $305° \Rightarrow \frac{9}{5}(\cos 305° + i \sin 305°)$

You have in your notes the proof for multiplying complex numbers in trigonometric form. Use a similar process to prove the following.

21.)

Given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $z_2 \neq 0$

,

prove that $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$.

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2(\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \cdot \frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)} \cdot \frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1\cos\theta_2 - \cos\theta_1 i\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2}{\cos^2\theta_2 - i^2\sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \cdot \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2}{1}$$

$$= \frac{r_1}{r_2} \cdot \underbrace{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2}_{\cos(u-v)} + i\underbrace{(\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2)}_{\sin(u-v)}$$

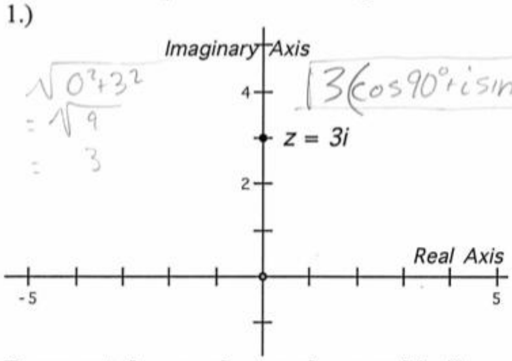
$$= \frac{r_1}{r_2} \cdot \cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)$$

Activity 5a - Products and Quotients of Complex Numbers in Trig Form

(Student Version)

Write the complex number in trigonometric form.

1.)



$\sqrt{0^2+3^2}$

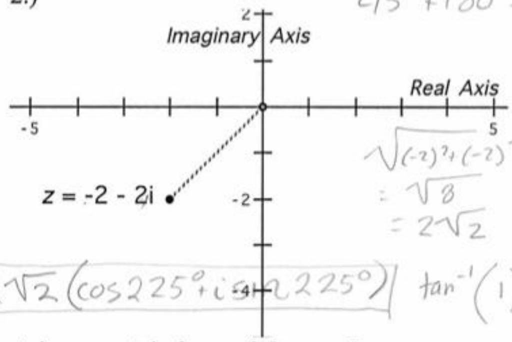
$=\sqrt{9}$

$= 3$

$\boxed{3(\cos 90^\circ + i\sin 90^\circ)}$

2.)

$45^\circ + 180^\circ = 225^\circ$



$\sqrt{(-2)^2+(-2)^2}$

$=\sqrt{8}$

$=2\sqrt{2}$

$\boxed{2\sqrt{2}(\cos 225^\circ + i\sin 225^\circ)}$ $\tan^{-1}(1)$

Represent the complex number graphically, and find the trigonometric form of the number.

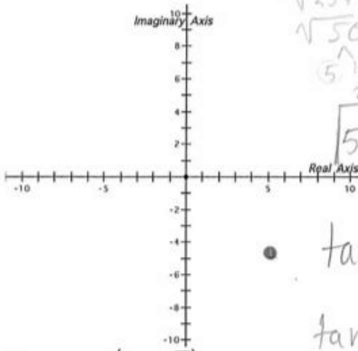
3.) $5-5i$

$\sqrt{(5)^2+(-5)^2}$

$\sqrt{25+25}$

$\sqrt{50} = 5\sqrt{2}$

$\boxed{5\sqrt{2}(\cos 315^\circ + i\sin 315^\circ)}$



$\tan^{-1}\left(\frac{-5}{5}\right)$

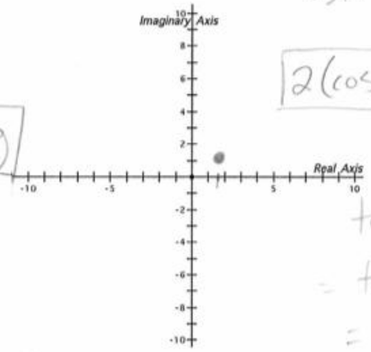
$\tan^{-1}(-1) =$

4.) $\sqrt{3}+i$

$\sqrt{(\sqrt{3})^2+1^2}$

$\sqrt{3+1} = \sqrt{4} = 2$

$\boxed{2(\cos 30^\circ + i\sin 30^\circ)}$



$\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)\left(\frac{\sqrt{3}}{\sqrt{3}}\right)$

$=\tan^{-1}\left(\frac{\sqrt{3}}{3}\right)$

$= 30^\circ$

5.) $-2\left(1+i\sqrt{3}\right)$

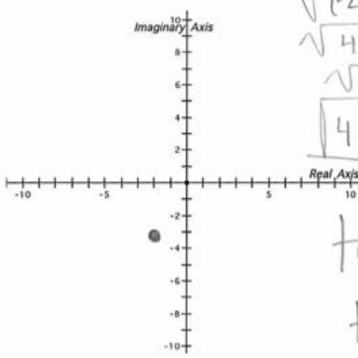
$-2-2i\sqrt{3}$

$\sqrt{(-2)^2+(-2\sqrt{3})^2}$

$\sqrt{4+12}$

$\sqrt{16} = 4$

$\boxed{4(\cos 240^\circ + i\sin 240^\circ)}$



$\tan^{-1}\left(\frac{\cancel{2}\sqrt{3}}{\cancel{2}}\right)$

$\tan^{-1}(\sqrt{3})$

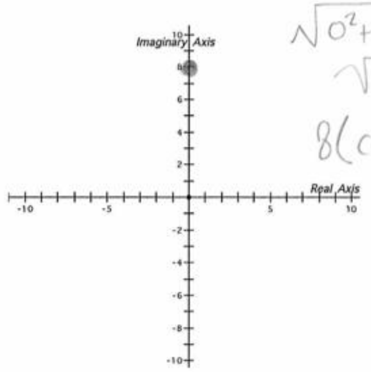
$60^\circ + 180^\circ = 240^\circ$

6.) $8i$

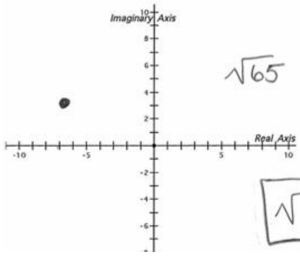
$\sqrt{0^2+8^2}$

$\sqrt{64} = 8$

$8(\cos 90^\circ + i\sin 90^\circ)$



7.) $-7+4i$ $\sqrt{(-7)^2+(4)^2} = \sqrt{49+16}$
$= \sqrt{65}$



$\sqrt{65}$ $\tan^{-1}\left(\frac{4}{-7}\right)$

$\sqrt{65}\,(\cos 5.76 + i\sin 5.76)$

$\boxed{\sqrt{65}\,(\cos 150.26^\circ + i\sin 150.26^\circ)}$

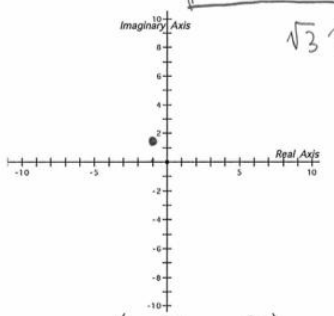
Represent the complex number graphically, and find the standard form of the number.

8.) $2(\cos 120^\circ + i\sin 120^\circ)$

$\cancel{2}\left(-\frac{1}{\cancel{2}}\right)\left(\frac{\sqrt{3}}{\cancel{2}}\right)$

$\boxed{-1 + i\sqrt{3}}$

$\sqrt{3} \approx 1.73$



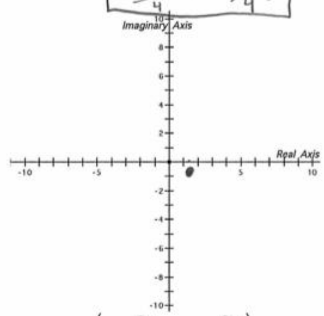
9.) $\frac{3}{2}(\cos 330^\circ + i\sin 330^\circ)$

$\frac{3}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{3}{2}\left(-\frac{1}{2}\right)$

$\boxed{\frac{3\sqrt{3}}{4} + -\frac{3}{4}i}$

$\frac{3\sqrt{3}}{4} \approx 1.3$

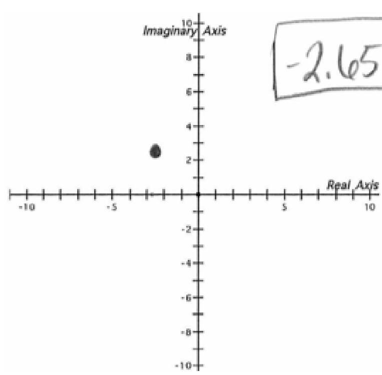
$-3/4 = -.75$



10.) $3.75\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$

$3.75\left(-\frac{\sqrt{2}}{2}\right)$ $3.75\left(\frac{\sqrt{2}}{2}\right)$

$-2.65 + 2.65i$



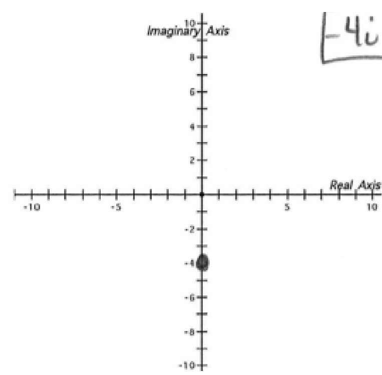
$\boxed{-2.65 + 2.65i}$

11.) $4\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$

0 -1

$0 + 4(-1)i$

$\boxed{-4i}$



$\boxed{-4i}$

Perform the operation and leave the result in trigonometric form.

12.) $\left[3\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]\left[4\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]$

$12\left[\cos\left(\frac{\pi}{3}+\frac{\pi}{6}\right)+i\sin\left(\pi/3+\pi/6\right)\right]$

$2\pi/6+\pi/6=3\pi/6=\pi/2$

$\boxed{12\left(\cos\pi/2+i\sin\pi/2\right)}=12i$

$0 \quad 1$

13.) $\frac{36}{1}$ $\left[\frac{3}{2}\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\right]\left[6\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)\right]$

$9\left[\cos\left(\frac{\pi}{6}+\frac{\pi}{4}\right)+i\sin\left(\pi/6+\pi/4\right)\right]$

$\frac{2\pi}{12}+\frac{3\pi}{12}$

$\frac{5\pi}{12}$

$\boxed{9\left(\cos 5\pi/12+i\sin\frac{5\pi}{12}\right)}$

14.) $\frac{2}{3}$ $\left[\frac{5}{3}(\cos140°+i\sin140°)\right]\left[\frac{2}{3}(\cos60°+i\sin60°)\right]$

$\frac{10}{9}\left[\cos(140°+60°)+i\sin(140°+60°)\right]$

$\frac{10}{9}\left[\cos200°+i\sin300°\right]$

$\boxed{\frac{10}{9}\left(\cos200°+i\sin200°\right)}$

15.) $(\cos5°+i\sin5°)(\cos20°+i\sin20°)$

$\left[\cos(5°+20°)+i\sin(5°+20°)\right]$

$\boxed{\cos25°+i\sin25°}$

16.) $\dfrac{\cos50°+i\sin50°}{\cos20°+i\sin20°}$

$\left[\cos(50°-20°)+i\sin(50°-20°)\right]$

$\boxed{\cos30°+i\sin30°}$

17.) $\dfrac{2(\cos120°+i\sin120°)}{4(\cos40°+i\sin40°)}$

$\frac{1}{2}\left[\cos(120°-40°)+i\sin(120°-40°)\right]$

$\boxed{\frac{1}{2}\left(\cos80°+i\sin80°\right)}$

18.) $\dfrac{\cos\left(\frac{7\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}\right)}{\cos\pi+i\sin\pi}$

$\left[\cos\left(\frac{7\pi}{4}-\frac{4\pi}{4}\right)+i\sin\left(\frac{7\pi}{4}-\pi\right)\right]$

$\boxed{\cos 3\pi/4+i\sin 3\pi/4}$

19.) $\dfrac{18(\cos54°+i\sin54°)}{3(\cos102°+i\sin102°)}$

$6\left[\cos(54°-102°)+i\sin(54°-102°)\right]$

$6\left[\cos(-48°)+i\sin(-48°)\right]$

$+360° \qquad +360°$

$\boxed{6\left(\cos312°+i\sin312°\right)}$

20.) $\dfrac{9(\cos20°+i\sin20°)}{5(\cos75°+i\sin75°)}=\frac{9}{5}\left[\cos(20°-75°)+i\sin(20°-75°)\right]$

$=\frac{9}{5}\left(\cos-55°+i\sin-55°\right)$ $\quad -55+360=305$

$=\boxed{\frac{9}{5}\left(\cos305°+i\sin305°\right)}$

You have in your notes the proof for multiplying complex numbers in trigonometric form. Use a similar process to prove the following.

21.) Given two complex numbers $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$ and $z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$, $z_2 \neq 0$,

prove that $\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]$.

$$\frac{z_1}{z_2} = \frac{r_1(\cos\theta_1 + i\sin\theta_1)}{r_2((\cos\theta_2 + i\sin\theta_2)}$$

$$= \frac{r_1}{r_2}\frac{(\cos\theta_1 + i\sin\theta_1)}{(\cos\theta_2 + i\sin\theta_2)}\frac{(\cos\theta_2 - i\sin\theta_2)}{(\cos\theta_2 - i\sin\theta_2)}$$

$$-(1)$$

$$= \frac{r_1}{r_2}\frac{\cos\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2 + i\sin\theta_1\cos\theta_2 - i^2\sin\theta_1\sin\theta_2}{\cos\theta_2^{\,2} \cancel{- i\sin\theta_2\cos\theta_2 + i\sin\theta_2\cos\theta_2} + i^2\sin\theta_2^{\,2}}$$

$$+(1)$$

$$= \frac{r_1}{r_2}\frac{\qquad}{\cos\theta_2^{\,2} + \sin\theta_2^{\,2}}$$

$$= \frac{r_1}{r_2}\left(\frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2) + i\sin\theta_1\cos\theta_2 - i\cos\theta_1\sin\theta_2}{1}\right.$$

$$= \frac{r_1}{r_2}\cos(\theta_1 - \theta_2) + i(\underbrace{\sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2}_{\sin(\theta_1 - \theta_2)})$$

$$= \frac{r_1}{r_2}\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)$$

$$\boxed{\frac{z_1}{z_2} = \frac{r_1}{r_2}\left[\cos(\theta_1 - \theta_2) + i\sin(\theta_1 - \theta_2)\right]}$$

Activity 5b SMART Board Slides – De Moivre's Theorem and Nth Roots

Slide 1

Trigonometric Form of a Complex Number
PART TWO

What you should learn:

How to use De Moivre's Theorem

How to find the nth roots of complex numbers

Slide 2

Powers of Complex Numbers

Given $z = r(\cos\theta + i\sin\theta)$

and applying the rules of multiplication we have

$$z^2 = r(\cos\theta + i\sin\theta)r(\cos\theta + i\sin\theta) = r^2(\cos 2\theta + i\sin 2\theta)$$

$$z^3 = r^2(\cos 2\theta + i\sin 2\theta)r(\cos\theta + i\sin\theta) = r^3(\cos 3\theta + i\sin 3\theta)$$

$$z^4 = r^3(\cos 3\theta + i\sin 3\theta)r(\cos\theta + i\sin\theta) = r^4(\cos 4\theta + i\sin 4\theta)$$

$$z^5 = r^4(\cos 4\theta + i\sin 4\theta)r(\cos\theta + i\sin\theta) = r^5(\cos 5\theta + i\sin 5\theta)$$

Which leads to

De Moivre's Theorem

If $z = r(\cos\theta + i\sin\theta)$ is a complex number and n is a positive integer, then

$$z^n = \left[r(\cos\theta + i\sin\theta)\right]^n$$
$$= r^n(\cos n\theta + i\sin n\theta)$$

Slide 3

Use De Moivre's Theorem to find the indicated power of the complex number. Express your result in standard form.

$$\left[2\left(\cos\frac{\pi}{10}+i\sin\frac{\pi}{10}\right)\right]^5 \qquad (2+2i)^6$$

Slide 4

The Fundamental Theorem of Algebra

The following is called the Fundamental Theorem of Algebra:

A polynomial of degree n has at least one root, real or complex.

This apparently simple statement allows us to conclude:

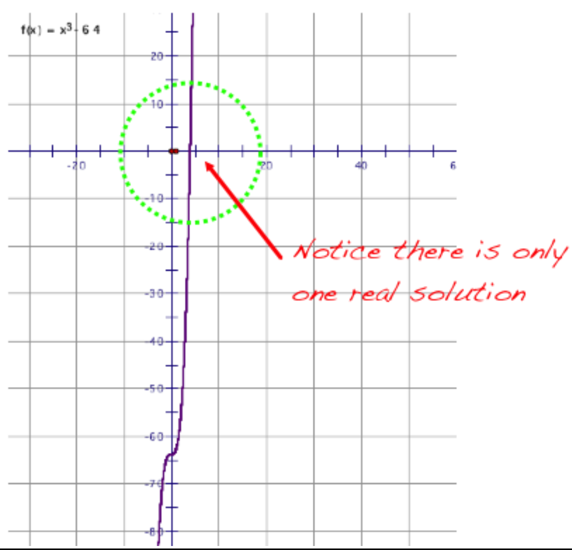
A polynomial P(x) of degree n has <u>exactly n roots</u>, real or complex.

If the leading coefficient of $P(x)$ is 1, then the Factor Theorem allows us to conclude:

$$P(x) = (x - r_n)(x - r_{n-1}) \ldots (x - r_2)(x - r_1)$$

This means for any polynomial we should be able to find n solutions or roots.

$$x^3 = 64$$



We rely on De Moivre's Theorem to help us find these roots.

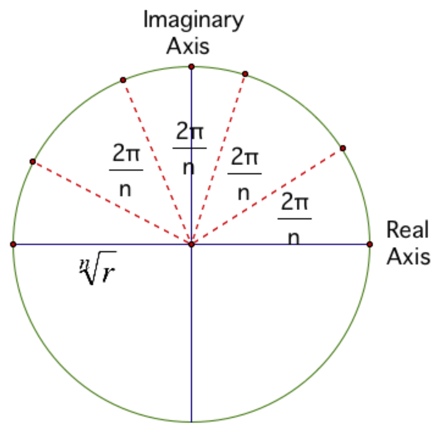
Slide 5

The Complex Root Theorem

For a positive integer n, the complex number $z = r(\cos\theta + i\sin\theta)$ has exactly n distinct roots given by

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

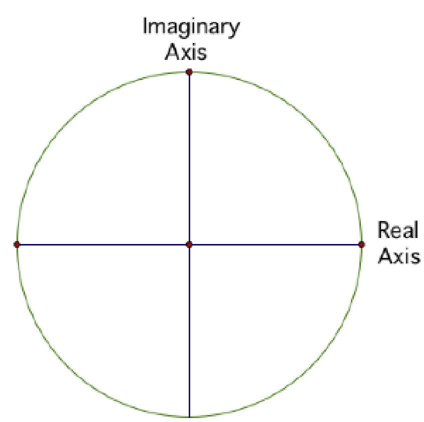
where k = 0, 1, 2, . . ., n - 1.



Slide 6

$$\sqrt[n]{r}\left(\cos\frac{\theta + 2\pi k}{n} + i\sin\frac{\theta + 2\pi k}{n}\right)$$

Find the third roots of 64.



Slide 7

$$\sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n}+i\sin\frac{\theta+2\pi k}{n}\right)$$

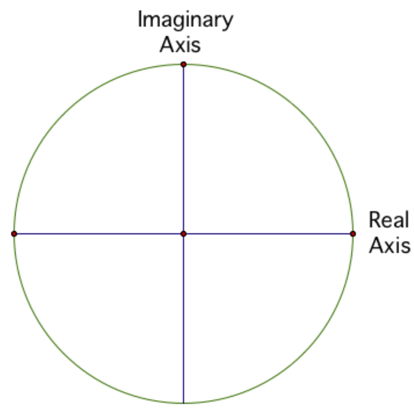
Find the square roots of $5(\cos 120^\circ + i\sin 120^\circ)$.

Find the fourth roots of -4.

Slide 8

$$\sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n}+i\sin\frac{\theta+2\pi k}{n}\right)$$

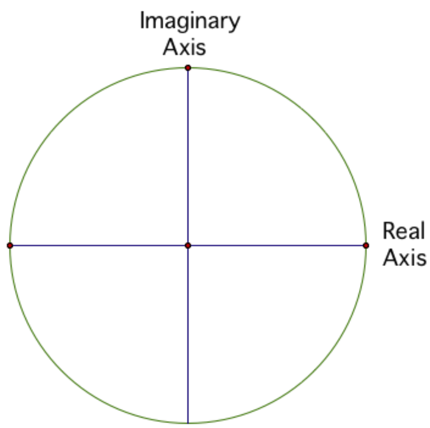
Find all of the solutions to the equation $x^4+81=0$.



Slide 9

The nth Roots of Unity
Find the fourth roots of 1.

$$\sqrt[n]{r}\left(\cos\frac{\theta+2\pi k}{n}+i\sin\frac{\theta+2\pi k}{n}\right)$$



Slide 10

Assignment: De Moivre's Theorem worksheet

Activity 5b – De Moivre's Theorem and Nth Roots

Use De Moivre's Theorem to find the indicated power of the complex number. Express the result in standard form.

1.) $(1+i)^3$

2.) $(-1+i)^{10}$

3.) $2\left(\sqrt{3}+i\right)^5$

4.) $(3-2i)^5$

5.) $\left[5(\cos 20° + i\sin 20°)\right]^3$

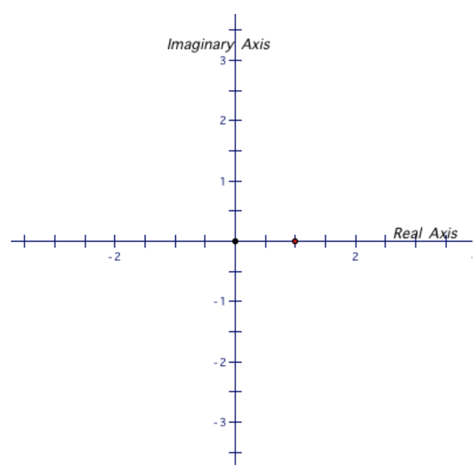
6.) $\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)^{10}$

7.) $\left[3(\cos 150° + i\sin 150°)\right]^4$

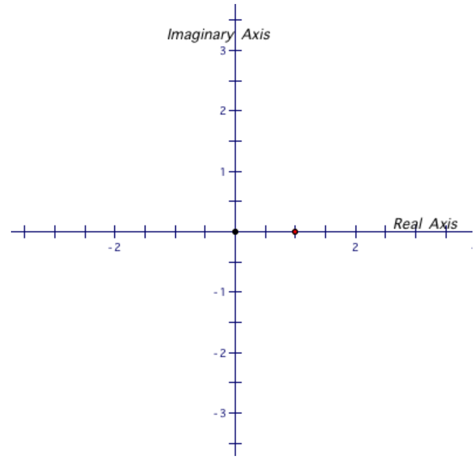
8.) $\left[4(\cos 2.8 + i\sin 2.8)\right]^5$

Use the Complex Root Theorem to find the indicated roots of the complex number and then represent each of the roots graphically. Express the roots in standard form.

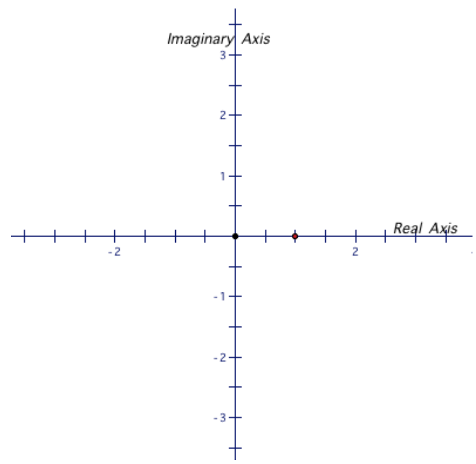
9.) Fourth roots of $16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$



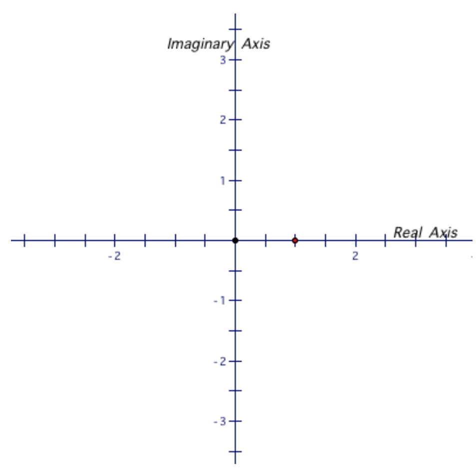
10.) Fifth roots of $32\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$



11.) Fourth roots of i

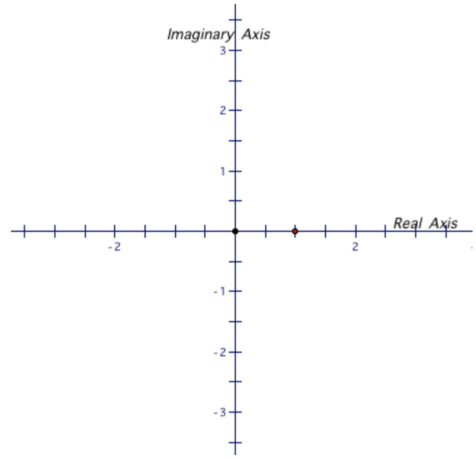


12.) Fifth roots of 1

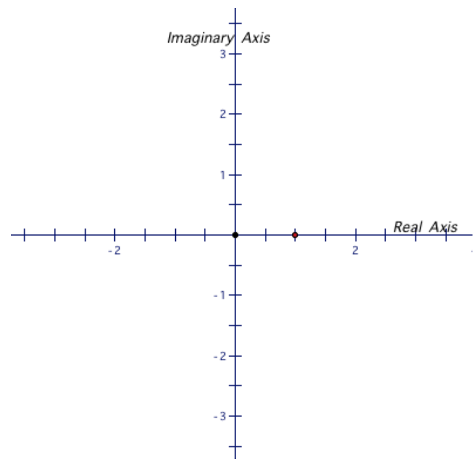


Use the Complex Roots Theorem to find all the solutions of the equation and represent the solutions graphically.

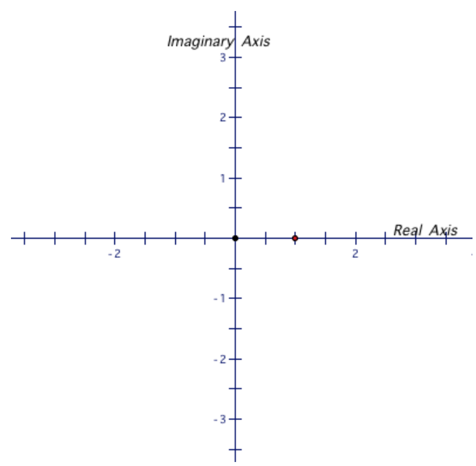
13.) $x^4 - i = 0$



14.) $x^5 - 243 = 0$



15.) $x^3 + 64i = 0$



Activity 5b – De Moivre's Theorem and Nth Roots (Teacher Version)

Use De Moivre's Theorem to find the indicated power of the complex number. Express the result in standard form.

1.) $(1+i)^3$ $\quad r=\sqrt{1^2+1^2}=\sqrt{2}$ $\quad \theta=\tan^{-1}(1)=\frac{\pi}{4}$

$z^3=\sqrt{2}\left(\cos\frac{\pi}{4}+i\sin\frac{\pi}{4}\right)^3$

$=2\sqrt{2}\left(\cos\frac{3\pi}{4}+i\sin\frac{3\pi}{4}\right)=-2+2i$

2.) $(-1+i)^{10}$

$=-32i$

3.) $2\left(\sqrt{3}+i\right)^5$

$=-32\sqrt{3}+32i$

4.) $(3-2i)^5$

$=-240-122i$

5.) $\left[5(\cos 20^\circ + i\sin 20^\circ)\right]^3$

$=62.5+62.5i\sqrt{3}$

6.) $\left(\cos\frac{5\pi}{4}+i\sin\frac{5\pi}{4}\right)^{10}$

$=i$

7.) $\left[3(\cos 150^\circ + i\sin 150^\circ)\right]^4$

$=-40.5-40.5i\sqrt{3}$

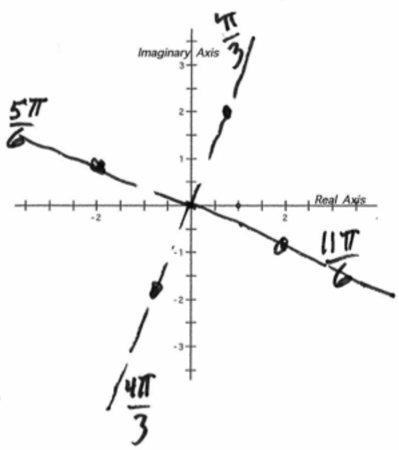
8.) $\left[4(\cos 2.8 + i\sin 2.8)\right]^5$

$\approx 140.02+1014.38i$

Use the Complex Root Theorem to find the indicated roots of the complex number and then represent each of the roots graphically. Express the roots in standard form.

9.) Fourth roots of $16\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$

Use $2\left(\cos\frac{\frac{4\pi}{3}+2\pi k}{4}+i\sin\frac{\frac{4\pi}{3}+2\pi k}{4}\right)$



k=0 $\quad 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$

k=1 $\quad 2\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

k=2 $\quad 2\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)$

k=3 $\quad 2\left(\cos\frac{11\pi}{6}+i\sin\frac{11\pi}{6}\right)$

k=0
$1+i\sqrt{3}$

k=1
$-\sqrt{3}+i$

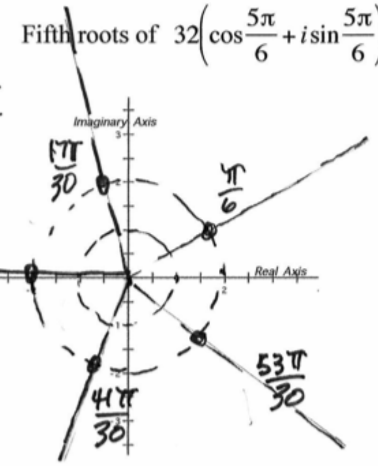
k=2
$-1-i\sqrt{3}$

k=3
$\sqrt{3}-i$

Use $2\left(\cos\frac{\frac{5\pi}{6}+2\pi k}{5}+i\sin\frac{\frac{5\pi}{6}+2\pi k}{5}\right)$

10.) Fifth roots of $32\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

$k=0$
$z_0=\sqrt{3}+i$
$k=1$
$z_1=-0.42+1.96i$
$k=2$
$z_2\,-1.99+0.21i$
$k=3$
$z_3=-0.81-1.83i$
$k=4$
$z_4=1.49-1.34i$



$k=0\quad 2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$

$k=1\quad 2\left(\cos\frac{17\pi}{30}+i\sin\frac{17\pi}{30}\right)$

$k=2\quad 2\left(\cos\frac{29\pi}{30}+i\sin\frac{29\pi}{30}\right)$

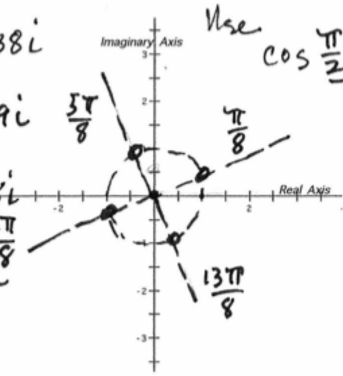
$k=3\quad 2\left(\cos\frac{41\pi}{30}+i\sin\frac{41\pi}{30}\right)$

$k=4\quad 2\left(\cos\frac{53\pi}{30}+i\sin\frac{53\pi}{30}\right)$

11.) Fourth roots of i $\quad z=i=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$

Use $\cos\frac{\frac{\pi}{2}+2\pi k}{4}+i\sin\frac{\frac{\pi}{2}+2\pi k}{4}$

$k=0$
$z_0=0.92+0.38i$
$k=1$
$z_1=-0.38+0.09i$
$k=2$
$z_2=-0.92-0.38i$
$k=3$
$z_3=0.38-0.92i$



$k=0\quad \cos\frac{\pi}{8}+i\sin\frac{\pi}{8}$

$k=1\quad \cos\frac{5\pi}{8}+i\sin\frac{5\pi}{8}$

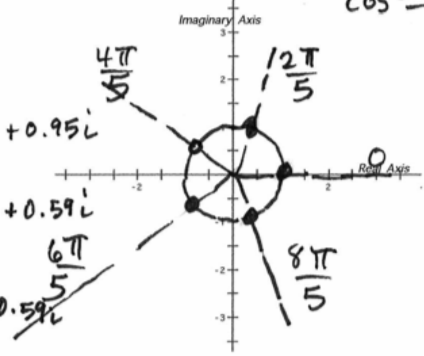
$k=2\quad \cos\frac{9\pi}{8}+i\sin\frac{9\pi}{8}$

$k=3\quad \cos\frac{13\pi}{8}+i\sin\frac{13\pi}{8}$

12.) Fifth roots of 1

Use $\cos\frac{0+2\pi k}{5}+i\sin\frac{0+2\pi k}{5}$

$k=0$
$z_0=1$
$k=1$
$z_1=0.31+0.95i$
$k=2$
$z_2=-0.81+0.59i$
$k=3$
$z_3=-0.81-0.59i$
$k=4$
$z_4=0.31-0.95i$



$k=0\quad \cos 0+i\sin 0$

$k=1\quad \cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}$

$k=2\quad \cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}$

$k=3\quad \cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}$

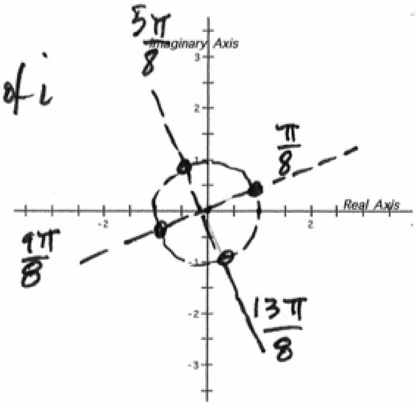
$k=4\quad \cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}$

Use the Complex Roots Theorem to find all the solutions of the equation and represent the solutions graphically.

13.) $x^4 - i = 0$ $\quad z = i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$

$x^4 = i$

4th roots of i



Use $\cos\dfrac{\frac{\pi}{2}+2\pi k}{4} + i\sin\dfrac{\frac{\pi}{2}+2\pi k}{4}$

$k=0 \quad \cos\frac{\pi}{8} + i\sin\frac{\pi}{8}$

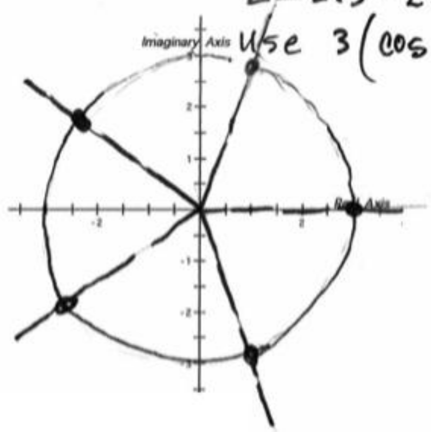
$k=1 \quad \cos\frac{5\pi}{8} + i\sin\frac{5\pi}{8}$

$k=2 \quad \cos\frac{9\pi}{8} + i\sin\frac{9\pi}{8}$

$k=3 \quad \cos\frac{13\pi}{8} + i\sin\frac{13\pi}{8}$

14.) $x^5 - 243 = 0$ $\quad z = 243 = 243(\cos 0 + i\sin 0)$

Use $3\left(\cos\frac{0+2\pi k}{5} + i\sin\frac{0+2\pi k}{5}\right)$



$k=0 \quad 3(\cos 0 + i\sin 0)$

$k=1 \quad 3\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$k=2 \quad 3\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$

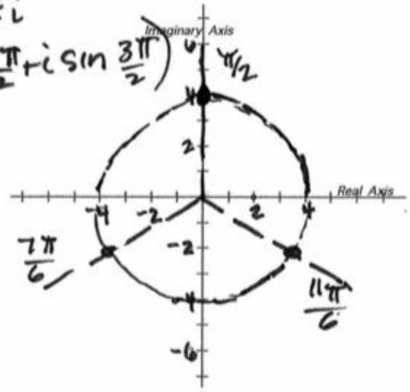
$k=3 \quad 3\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$

$k=4 \quad 3\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$

15.) $x^3 + 64i = 0$

$z = -64i$

$= 64\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right)$



$k=4 \quad 3\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$

use

$$4\left(\cos\frac{\frac{3\pi}{2}+2\pi k}{3} + i\sin\frac{\frac{3\pi}{2}+2\pi k}{3}\right)$$

$k=0 \quad 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$

$k=1 \quad 4\left(\cos\frac{7\pi}{6} + i\sin\frac{7\pi}{6}\right)$

$k=2 \quad 4\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$

Activity 5b - De Moivre's Theorem and Nth Roots (Student Work)

Use De Moivre's Theorem to find the indicated power of the complex number. Express the result in standard form.

1.) $(1+i)^3$

$r = \sqrt{1^2+1^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = 45° = \pi/4$

$\left[\sqrt{2}\left(\cos \pi/4 + i\sin \pi/4\right)\right]^3$

$\sqrt{2}^3\left(\cos 3\pi/4 + i\sin 3\pi/4\right)$

$\sqrt{2}^3\left(-\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}i\right) = \boxed{-2+2i}$

2.) $(-1+i)^{10}$

$r = \sqrt{(-1)^2+(1)^2} = \sqrt{2}$

$\theta = \tan^{-1}\left(\frac{1}{1}\right) = -45$, $+180$, $135°$

$\left[\sqrt{2}\left(\cos 3\pi/4 + i\sin 3\pi/4\right)\right]^{10}$

$\sqrt{2}^{10}\left(\cos \frac{30\pi}{4} + i\sin \frac{30\pi}{4}\right)$

$32\left(\cos \frac{15\pi}{2} + i\sin \frac{15\pi}{2}\right) \rightarrow 32\left(\cos \frac{3\pi}{2} + i\sin \frac{3\pi}{2}\right) = \boxed{-32i}$

3.) $2\left(\sqrt{3}+i\right)^5$

$r = \sqrt{(\sqrt{3})^2+1^2} = \sqrt{3+1} = \sqrt{4} = 2$

$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = 30° = \pi/6$

$2\left[2^5\left(\cos \pi/6 + i\sin \pi/6\right)\right]$

$2\left[32\left(\frac{\sqrt{3}}{2} + i\frac{1}{2}\right)\right] \rightarrow 32\left[\sqrt{3}+i\right] = \boxed{32\sqrt{3}+32i}$

4.) $(3-2i)^5$

$r = \sqrt{3^2+(-2)^2} = \sqrt{9+4} = \sqrt{13}$

$\theta = \tan^{-1}\left(\frac{-2}{3}\right) \approx 326.31$ ($-33.6+360$)

$\sqrt{13}^5\left(\cos^5 326.31° + i\sin^5 326.31°\right)$

$\sqrt{13}^5\left(\cos 1631.55 + i\sin 1631.55\right)$

$(609.3)(.97) + (609.3)(.20) \rightarrow \boxed{-596.9-122.0i}$

5.) $\left[5(\cos 20° + i\sin 20°)\right]^3$

$= \left[5(\cos 20° + i\sin 20°)\right]^3$

$= 5^3\left(\cos 3(20°) + i\sin 3(20°)\right)$

$= 125\left(\cos 60° + i\sin 60°\right) = 125\left(\frac{1}{2} + i\frac{\sqrt{3}}{2}\right) = \boxed{\frac{125}{2} + \frac{125\sqrt{3}i}{2}}$

6.) $\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)^{10}$

$1^{10}\left(\cos 10\left(\frac{5\pi}{4}\right) + i\sin 10\left(\frac{5\pi}{4}\right)\right)$

$\left(\cos \frac{25\pi}{2} + i\sin \frac{25\pi}{2}\right)$

$\left(\cos \pi/2 + i\sin \pi/2\right)$

$0 + i\cdot 1 = \boxed{i}$

7.) $\left[3(\cos 150° + i\sin 150°)\right]^4$

$3^4\left(\cos 4\cdot 150 + i\sin 4\cdot 150\right)$

$81\left(\cos 600 + i\sin 600\right)$ $\quad 600-360$

$81\left(\cos 240 + i\sin 240\right)$

$81 \;\; -\frac{1}{2} \;\; -\frac{\sqrt{3}}{2}i = \boxed{-\frac{81}{2} - \frac{81i\sqrt{3}}{2}}$

8.) $\left[4(\cos 2.8 + i\sin 2.8)\right]^5$

$4^5\left[\cos 5(2.8) + i\sin 5(2.8)\right]$

$1024\left[\cos 14 + i\sin 14\right]$

$1024\left[\cos 14 + i\sin 14\right]$

$1024(.136 + i\;.99) \rightarrow 140.02 + 1014.38i$

$\boxed{140.02+1014.4i}$

Use the Complex Root Theorem to find the indicated roots of the complex number and then represent each of the roots graphically. Express the roots in standard form.

9.) Fourth roots of $16\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right)$

$\sqrt[4]{16} = 2 \quad \left(\cos \frac{4\pi/3 + 2\pi(0)}{4} + i\sin \frac{4\pi/3 + 2\pi(0)}{4}\right)$

$2\left(\cos \frac{4\pi}{3}\cdot\frac{1}{4}\right) \rightarrow 2\left(\cos \pi/3\right)$

$2\left(\cos \pi/3 + i\sin \pi/3\right)$

$2\left(\frac{1}{2}\right) \; \boxed{1 + \sqrt{3}i}$

$2\left(\cos \frac{4\pi/3 + 2\pi}{4} + i\sin 5\pi/6\right) \rightarrow 2\left(\cos 5\pi/6 + i\sin 5\pi/6\right)$

$2\left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$

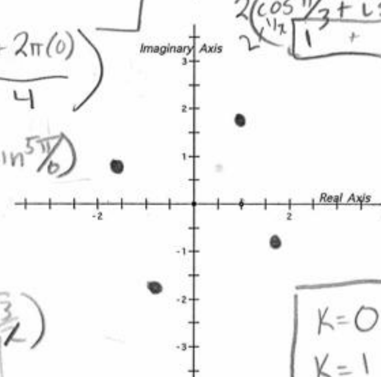
$\boxed{-\sqrt{3}+i}$

$2\left(\cos \frac{4\pi/3 + 4\pi}{4} + i\sin 4\pi/3\right) \rightarrow 2\left(-\frac{1}{2} + i\frac{-\sqrt{3}}{2}\right)$

$\boxed{-1-i\sqrt{3}}$

$2\left(\cos \frac{4\pi/3 + 6\pi}{4} + i\sin \frac{11\pi}{6}\right) \rightarrow 2\left(\frac{\sqrt{3}}{2} - \frac{1}{2}i\right) =$

$\boxed{\sqrt{3}-i}$



K=0	$1+\sqrt{3}i$
K=1	$-\sqrt{3}+i$
K=2	$-1-i\sqrt{3}$
K=3	$\sqrt{3}-i$

$K=0$	$\sqrt{3}+i$
$K=1$	$-.42+1.9i$
$K=2$	$-1.99+.21i$
$K=3$	$-.81-1.83i$
$K=4$	$1.49-1.34i$

10.) Fifth roots of $32\left(\cos\frac{5\pi}{6}+i\sin\frac{5\pi}{6}\right)$

$\sqrt[5]{32}=2$

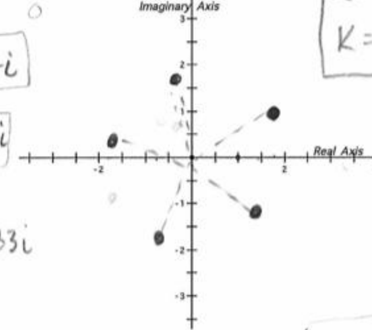
$2\left(\cos\frac{5\pi}{6}\cdot\frac{1}{5}+i\sin\frac{\pi}{6}\right)\to 2\left(\frac{\sqrt{3}}{2}+i\frac{1}{2}\right)=\boxed{\sqrt{3}+i}$

$2\left(\cos\frac{5\pi}{6}+\frac{12\pi}{6}=\frac{17\pi}{6\cdot 5}=\frac{17\pi}{30}+i\sin\frac{17\pi}{30}\right)=\boxed{-.42+1.9i}$

$2\left(\cos\frac{29\pi}{30}+i\sin\frac{29\pi}{30}\right)=-1.99+.21i$

$2\left(\cos\frac{5\pi}{6}+\frac{36\pi}{6}=\frac{41\pi}{6\cdot 5}=\frac{41\pi}{30}+i\sin\frac{41\pi}{30}\right)=-.81-1.83i$

$2\left(\cos\frac{5\pi}{6}+\frac{48\pi}{6}=\frac{53\pi}{6\cdot 5}=\frac{53\pi}{30}+i\sin\frac{53\pi}{30}\right)=$



$K=0$	$.92+.38i$
$K=1$	$-.38+.92i$
$K=2$	$-.92-.38i$
$K=3$	$.38-.92i$

11.) Fourth roots of i $\quad z=i$

$\frac{\pi}{2}+\frac{16\pi}{2}=\frac{13\pi}{2}\cdot\frac{1}{4}=\frac{13\pi}{8}$

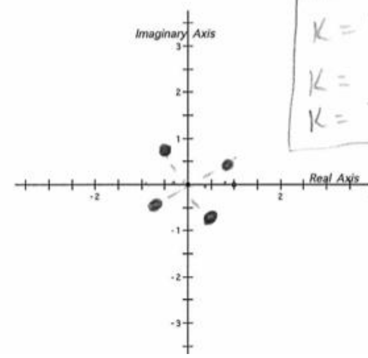
$$\frac{\cos\frac{\pi}{2}+2\pi K}{4}+\frac{i\sin\frac{\pi}{2}+2\pi K}{4}$$

$K=0\quad \cos\frac{\pi}{8}+i\sin\frac{\pi}{8}=.92+.38i$

$K=1\quad \cos\frac{5\pi}{8}+i\sin\frac{5\pi}{8}=-.38+.92i$

$K=2\quad \cos\frac{9\pi}{8}+i\sin\frac{9\pi}{8}=-.92-.38i$

$K=3\quad \cos\frac{13\pi}{8}+i\sin\frac{13\pi}{8}=.38-.92i$



12.) Fifth roots of 1 $\quad 1+0i$

$K=0\quad \cos 0+i\sin 0=1+0i$

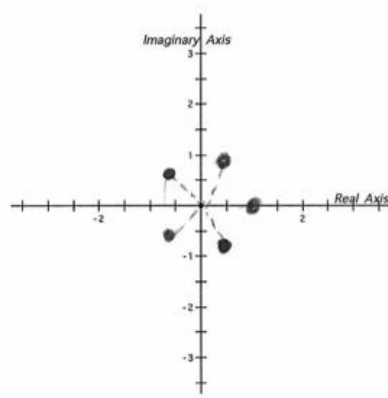
$K=1\quad \cos\frac{2\pi}{5}+i\sin\frac{2\pi}{5}=.31+.95i$

$K=2\quad \cos\frac{4\pi}{5}+i\sin\frac{4\pi}{5}=-.81+.59i$

$K=3\quad \cos\frac{6\pi}{5}+i\sin\frac{6\pi}{5}=-.81-.59i$

$K=4\quad \cos\frac{8\pi}{5}+i\sin\frac{8\pi}{5}=.31-.95i$

$$\cos\frac{0+2\pi K}{5}+i\sin\frac{0+2\pi K}{5}$$



$K=0$	1
$K=1$	$.31+.95i$
$K=2$	$-.81+.59i$
$K=3$	$-.81-.59i$
$K=4$	$.31-.95i$

Use the Complex Roots Theorem to find all the solutions of the equation and represent the soluti graphically.

13.) $x^4 - i = 0$

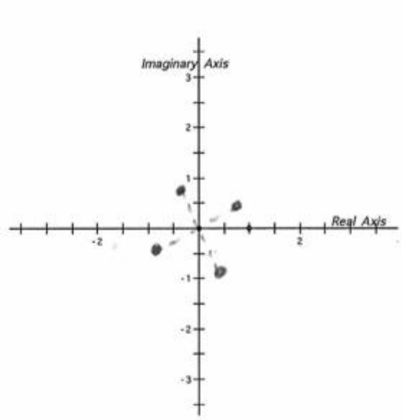
$x^4 = i \to x = \sqrt[4]{i} \quad z = i$

$K_0 = \cos \pi/8 + i\sin \pi/8 = .92 + .38i \quad K_0$

$K_1 = \cos 5\pi/8 + i\sin 5\pi/8 = -.38 + .92i = K_1$

$K_2 = \cos 9\pi/8 + i\sin 9\pi/8 = -.92 - .38i = K_2$

$K_3 = \cos 13\pi/8 + i\sin 13\pi/8 = .38 - .92i = K_3$



14.) $x^5 - 243 = 0 \quad x^5 = 243$

$x = \sqrt[5]{243} = 3$

$K_0 = 3(\cos 0 + i\sin 0) = 3$

$K_1 = 3(\cos 2\pi/5 + i\sin 2\pi/5)$

$K_2 = 3(\cos 4\pi/5 + i\sin 4\pi/5)$

$K_3 = 3(\cos 6\pi/5 + i\sin 6\pi/5)$

$K_4 = 3(\cos 8\pi/5 + i\sin 8\pi/5)$

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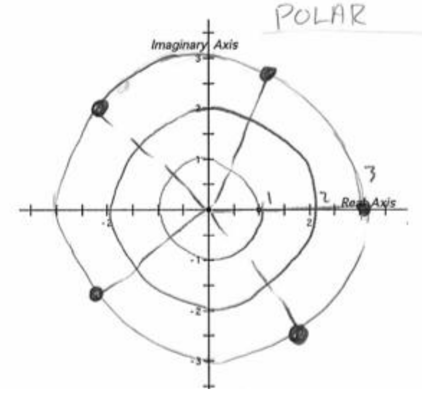


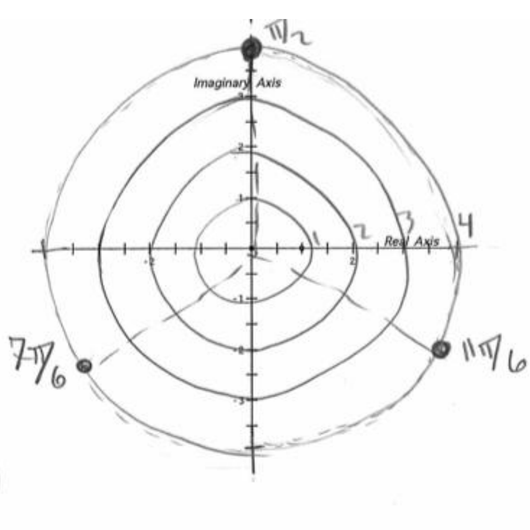
15.) $x^3 + 64i = 0 \quad x^3 = -64i \quad z = -64i$

$k = 0 \quad \sqrt[3]{64} = 4$

$K_0 = 4(\cos \frac{3\pi}{2} \cdot \frac{1}{3}) \to 4(\cos \pi/2 \quad i\sin \pi/2)$

$K_1 = 4(\cos 7\pi/6 + i\sin 7\pi/6)$

$K_2 = 4(\cos 11\pi/6 + i\sin 11\pi/6)$



$4\left(\cos \frac{\frac{3\pi}{2} + 2\pi k}{n}\right) + i\left(\sin \frac{3\pi/2 + 2\pi k}{n}\right)$

$3\pi + 4$

$7\pi/2 \cdot 3 = 7\pi/6 \qquad \frac{3\pi}{2} + \frac{4\pi}{2} = \frac{11\pi}{2} \cdot 3 = 11\pi/6$

Reflection on Activities 5a and 5b

Students had some difficulty with the process of re-writing complex numbers in trig form, particularly with values in quadrants 2 – 4. Students were not confident with their knowledge of the range of the inverse tangent function. They were reluctant to plot an imaginary number and then use the location of that point to visualize an approximate angle of rotation from the positive x-axis. When faced with an angle from the tangent inverse that seemed incorrect students were confused and unsure whether to add 2π or π in order to obtain the angle, which terminates in the 2nd or 3rd quadrant. With repeated instructions to consider which quadrant the rectangular coordinates would lie in, and reminders of the restrictions to the range of the inverse tangent function, students got better at conversions.

Students had little trouble with the process of multiplying and dividing once a complex number was converted to trig form. Students had little trouble with the process of De Moivre's Theorem. They enjoyed having a series of clearly defined steps to work through, rather than the ambiguity of proving trigonometric identities. There was very little interest in the proof of this theorem. Students found it tedious and were reluctant to follow along.

The process of finding roots was almost as easy for students as powers. I expected students to have more difficulty with finding nth roots of equations such as $x^4 - i = 0$. Most students were quick to find the first root with the formula for roots and then found multiples of the angle for this value based on the exponent for x. Several students struggled with writing the complex number in trig form if there was no imaginary part. For example if the equation to solve was $x^5 - 243 = 0$ they had trouble visualizing the point on the complex plane.
Student learning would have improved greatly with an additional class period of lecture and practice.

Final Reflection

While it was with some trepidation the curriculum for this class was re-ordered and supplemented, it was satisfying to observe students identify similarities between plotting complex numbers and vectors. It was surprising to find that with practice, students were quite capable of writing complex numbers in trigonometric form. Students continued to grumble regarding the need for another number system throughout the lessons, until they began to find nth roots of equations. Here not every student came to see the purpose for complex numbers but many did, and a handful even seemed to grasp the elegance of our number system where rules stay true and values hold despite the addition of an imaginary unit.

In changing individual lessons it seems better to combine operations of complex numbers along with graphing on the complex plane. However, it would be necessary to review these concepts along with instruction on the absolute value of complex numbers prior to writing complex numbers in polar or trig form. The textbook gives very little precedent for writing complex numbers in polar form and the section on polar numbers comes four chapters later. Students are likely to question what they are being taught. It was helpful to provide students with a more thorough understanding of polar coordinates before turning to complex numbers in their polar or trigonometric form.

Finally, if there were one over-arching element to change for these lessons on complex numbers, De Moivre's Theorem, and nth roots, it would be to have more time for each lesson and more time for student practice. Students felt pushed to perform without feeling confident in their understanding. There was not time to introduce any of the history of complex numbers, Abraham De Moivre, or Euler's formula $e^{ix} = \cos x + i\sin x$. Certainly these are important to student learning. Student disbelief at the need for an additional number system was an ongoing topic of conversation throughout these lessons. Perhaps insight would be gained through learning even some of the history of the complex number system and the great mathematicians that contributed to its discovery.

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